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Symmetries of differential equations. IV

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By an application of the geometrical techniques of Lie, Cohen, and Dickson it is shown that a system of differential equations of the form $x_i^{(r_i} = F_i$ (where $r_i > 1$ for every $i = 1,...,n$) cannot admit an infinite number of pointlike symmetry vectors. When $r_i = r$ for every $i = 1,...,n$, upper bounds have been computed for the maximum number of independent symmetry vectors that these systems can possess: The upper bounds are given by $2n^2 + nr + 2$ (when $r > 2$), and by $2n^2 + 4n + 2$ (when $r = 2$). The group of symmetries of $\bar{x}^{(r} = \bar{0}$ $(r > 1)$ has also been computed, and the result obtained shows that when $n > 1$ and $r > 2$ the number of independent symmetries of these equations does not attain the upper bound $2n^2 + nr + 2$, which is a common bound for all systems of differential equations of the form $\bar{x}^{(r} = \bar{F}(t,\bar{x},...,\bar{x}^{(r-1})$ when $r > 2$. On the other hand, when $r = 2$ the first upper bound obtained has been reduced to the value $n^2 + 4n + 3$; this number is equal to the number of independent symmetry vectors of the system $\ddot{\bar{x}} = \bar{0}$, and is also a common bound for all systems of the form $\ddot{\bar{x}} = \bar{F}(t,\bar{x},\dot{\bar{x}})$.

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I. INTRODUCTION

This paper should be considered as a continuation of a series of papers by the authors,[1] in this and other journals, on the fascinating subject of the symmetries of systems of differential equations. In these papers both the direct and the inverse problem concerning the symmetries have been studied, as well as certain connections between the symmetry vectors and the first integrals of systems of differential equations. Although some global results have been obtained, most of the results obtained are of a local character.

In the present paper we obtain, following the geometrical and local techniques contained in the classical treatises of Lie and Scheffers, Cohen, and Dickson,[2] upper bounds for the number of independent pointlike symmetry vectors of differential equations of the form

$$\bar{x}^{(r} = \bar{F}(t,\bar{x},...,\bar{x}^{(r-1}), \tag{i}$$

where $r > 1$ and $\bar{x}$ stands for $(x_1,...,x_n)$. The case $r = 1$ has not been studied, since it is well known—see, for instance, the first and fourth papers quoted in Ref. 1—that when $r = 1$ the number of independent symmetries is always infinite.

We obtain in Sec. III the upper bound $2n^2 + nr + 2$ $(r > 2)$, as well as the number of independent symmetry vectors of the system $\bar{x}^{(r} = \bar{0}$, which is given by $n^2 + nr + 3$, and the explicit expression of them. Since $2n^2 + nr + 2$ is greater than $n^2 + nr + 3$ when $n > 1$, the problem arises of knowing whether or not the upper bound $2n^2 + nr + 2$ is attained by a system of differential equations of this type, when $n > 1$.

Similarly, for a system of the form $\ddot{\bar{x}} = \bar{F}(t,\bar{x},\dot{\bar{x}})$, we obtain in Sec. IV the upper bound $2n^2 + 4n + 2$, which is reduced in Sec. V to $n^2 + 4n + 3$ by using a remarkable property of the projective group. This last upper bound is attained by the system $\ddot{\bar{x}} = \bar{0}$, whose symmetry group is the projective group of pointlike transformations of the space $\{(t,\bar{x})\}$.

When $n = 1$, i.e., when only a single differential equation is considered, the upper bounds obtained reduce to $r + 4$ (when $r > 2$) and 8 (when $r = 2$). These two results are classical and well known, and the proof we give of them in Sec. II tries only to be a bit more careful than the classical proofs, at the same time preparing the reader for a clearer understanding of the more complicated case of a normal system of differential equations of the form

$$x_i^{(r_i} = F_i, \quad r_i > 1, \quad \forall i = 1,...,n. \tag{ii}$$

As is shown in Sec. VI, a system of this type possesses only a *finite* number $N(n;r_1,...,r_n)$ of independent symmetry vectors, and this number grows without limit when either n or some of the r_i's tend to infinity. The conclusion is that a system of differential equations of the type $x_i^{(r_i} = F_i$, with $r_i > 1$ for every i, does not admit a Lie group (in the generalized sense of a group of transformations with an *infinite* number of essential parameters) as its symmetry group.

The reader should consult the classical treatises cited in Refs. 2 and 5 for most of the definitions and the notation used here, as well as the first three papers of this series cited in Ref. 1.

II. MAXIMUM NUMBER OF INDEPENDENT SYMMETRY VECTORS OF A DIFFERENTIAL EQUATION OF ORDER $r > 1$

In order that the reader can follow us without difficulty in the more complicated case of a normal system of differential equations, it is convenient to treat first the relatively simple case of a single differential equation of the form

$$x^{(r} = F(t,x,\dot{x},...,x^{(r-1}). \tag{1}$$

We remind the reader that when $r = 1$ Eq. (1) always possesses an infinite number of independent symmetry vec-

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tors.[1] On the contrary, when $r > 1$ Eq. (1) does *not* admit, in general, pointlike families of symmetries of the form

$$t' = t + \epsilon \cdot a(t,x), \quad x' = x + \epsilon \cdot b(t,x). \tag{2}$$

In particular, when $r = 2$, one can even classify[1] all the differential equations of the form

$$\ddot{x} = F(t,x), \tag{3}$$

admitting at least one symmetry vector of type (2) different from zero.

Moreover, concerning the pointlike symmetry vectors, it is a classical result that when $r > 1$, Eq. (1) admits no more than eight symmetry vectors (if $r = 2$) and no more than $(r+4)$ if $r > 2$. The proof of this result, or at least the fundamental ideas behind it, can be found in the classical treatises of Lie, Cohen and Dickson.[2] For the sake of completeness, we present here a proof of this classical result, which tries to be a bit more careful than the one presented by the above-mentioned authors, and at the same time prepares the reader for the more complicated case of a normal system of differential equations of the following type:

$$x_i^{(r_i} = F_i, \quad r_i > 1, \quad \forall i = 1,\dots,n, \tag{4}$$

where the smooth functions F_i appearing in (4) depend, of course, on the variables $t;x_1,\dots,x_1^{(r_1-1};x_n,\dots,x_n^{(r_n-1}$.

We begin by studying the case $r > 2$:

(a) Consider the unique solution $\phi(t;\lambda)$ of (1) corresponding to the initial conditions $(t_0,x_0,\dots,x_0^{(r-2};\lambda)$, and let $P_1 = (t_1,x_1 = \phi(t_1))$, with t_1 sufficiently close to t_0, and

$$\phi(t) = \phi(t;x_0^{(r-1}) \tag{5}$$

for an arbitrary, but fixed, $x_0^{(r-1}$. We shall now show that for certain neighborhoods U_1 of P_1 and I_1 of $x_0^{(r-1}$ there exists a unique smooth (i.e., C^∞) function $\theta_1: U_1 \to I_1$ satisfying

(i) $\theta_1(P_1) = x_0^{(r-1}$;

(ii) If $P = (t,x) \in U_1$ and $x^{(r-1} \in I_1$, then

$$\phi(t;x^{(r-1}) = x \quad \text{iff}\ x^{(r-1} = \theta_1(P).$$

That is, through every point of U_1 there passes a *unique* integral curve of (1) whose $(r-1)$th derivative lies on I_1 having a contact of order $(r-2)$ at $P_0 = (t_0,x_0)$ with the integral curve γ_0 of (1) corresponding to the initial conditions $(t_0,x_0,\dots,x_0^{(r-1})$.

The proof follows from the fact that, regarded as functions of t and of the initial conditions $t_0,x_0,\dots,x_0^{(r-1}$, the solutions of (1) are C^∞ functions, provided only that the function F appearing in (1) is, as we shall assume throughout this paper, a C^∞ function of its variables. Therefore, $\phi(t;\lambda)$ will be also smooth in t and λ, and since the triplet $(t_1,x_1,x_0^{(r-1})$ satisfies the equation

$$x = \phi(t;\lambda), \tag{6}$$

in order to complete our proof, it suffices to show that for t_1 sufficiently close to t_0 the "transversality condition"

$$\left.\frac{\partial\phi}{\partial\lambda}\right|_{(t_1,x_0^{(r-1})} \neq 0 \tag{7}$$

holds; indeed, if this were the case, the implicit function theorem[3] applied to (6) in a neighborhood of the point $(t_1,x_1,x_0^{(r-1})$ would yield λ as a smooth function θ_1 of the variables t and x.

Now, one can obviously write

$$\phi(t;\lambda) = x_0 + \dot{x}_0(t-t_0) + \cdots + x_0^{(r-2}(t-t_0)^{r-2}/(r-2)! + \lambda(t-t_0)^{r-1}/(r-1)! + (t-t_0)^r R(t,\lambda), \tag{8}$$

$R(t,\lambda)$ being a C^∞ function of t and λ near $(t_0,x_0^{(r-1})$.[4]

Therefore, one can also write

$$\frac{\partial\phi}{\partial\lambda} = \frac{(t-t_0)^{r-1}}{(r-1)!} + (t-t_0)^r\frac{\partial R}{\partial\lambda} \tag{9}$$

and, accordingly,

$$\left.\frac{\partial\phi}{\partial\lambda}\right|_{(t_1,x_0^{(r-1})} = (t_1-t_0)^{r-1}\left[\frac{1}{(r-1)!} + (t_1-t_0)R_1\right] \tag{10}$$

where $R_1 = \left.\dfrac{\partial R}{\partial\lambda}\right|_{(t_1,x_0^{(r-1})}$.

This last expression guarantees that (7) holds provided only that one chooses $t_1 \neq t_0$ satisfying

$$|(t_1-t_0)\cdot R_1| < 1/(r-1)!, \tag{11}$$

which is possible since R is continuous (C^∞ in fact).

Summarizing, the implicit function theorem applied to (6) yields the unique smooth function θ_1 satisfying conditions (i) and (ii) above.

(b) Let now $\phi_1(t;\lambda)$ be the maximal solution of (1) corresponding to the initial conditions $(t_1,x_1,\dots,x_1^{(r-2};\lambda)$, where

$$x_1^{(k} = \phi^{(k}(t_1) \tag{12}$$

and $\phi(t)$ is the function defined by (5). Choosing now a third point P_2 on $\gamma_0 \cap U_1$ sufficiently close to P_1, and repeating the construction sketched in (a) with P_0 and P_1 replaced respectively by P_1 and P_2, we obtain a second function $\theta_2: U_2 \to I_2$ satisfying:

(a) $\theta_2(P_2) = x_1^{(r-1} \in I_2$;

(b) If $P = (t,x) \in U_2$ and $x^{(r-1} \in I_2$, then

$$\phi_1(t;x^{(r-1}) = x \quad \text{iff} \quad x^{(r-1} = \theta_2(P).$$

Since $U = U_1 \cap U_2 \neq \varnothing$ and $U \subset U_1$, the mapping $\theta: U \to I_1 \times I_2$ defined by

$$P \mapsto \theta(P) = (\theta_1(P),\theta_2(P)) \tag{13}$$

is such that, given any two integral curves of (1), $\gamma_1 = (t,f_1(t))$ and $\gamma_2 = (t,f_2(t))$, having a contact of order $(r-2)$ with γ_0, respectively, at P_0 and P_1 and satisfying

$$f_1^{(r-1}(t_0) \in I_1, \quad f_2^{(r-1}(t_1) \in I_2, \tag{14}$$

then γ_1 and γ_2 will pass through a point $P \in U$ if and only if

$$(f_1^{(r-1}(t_0), f_2^{(r-1}(t_1)) = \theta(P). \tag{15}$$

(c) Let now U_0 be an open subset of $(U_1 \cap U_2) - \gamma_0$: If $P \in U_0$, then P will be an isolated point of $\gamma_1 \cap \gamma_2$ [where γ_1 and γ_2 are, of course, the curves defined in (b) passing through P].

In fact, if this were not the case one could immediately write

$$f_1^{(k}(t_p) = f_2^{(k}(t_p), \quad k = 0,1,2,\cdots, \tag{16}$$

where $P = (t_P,x_P)$. Clearly, we can restrict ourselves to the case $t_0 < t_P < t_1$ $\forall P \in U_0$. We then define $f(t)$ as follows:

$$f(t) = \begin{cases} f_1(t) & \text{when } t \leqslant t_P, \\ f_2(t) & \text{when } t \geqslant t_P. \end{cases} \quad (17)$$

Then condition (16) guarantees that $f(t)$ is a C^∞ function in some interval $J \supset (t_0, t_1)$ such that

$$f^{(r-1}(t_0) = f_1^{(r-1}(t_0) \in I_1. \quad (18)$$

Therefore, the curve $\gamma = \{(t, f(t))\}$ is an integral curve of (1) having a contact of order $(r-2)$ with γ_0 at P_0 and passing through $P_1 \in U_1$: Hence (18) and the properties of the function θ_1 [see (i) and (ii) above] imply that

$$f^{(r-1}(t_0) = \theta_1(P_1) = x_0^{(r-1} = \phi^{(r-1}(t_0). \quad (19)$$

It follows, by uniticity, that $f(t) = \phi(t)$ and, in particular, $P \in \gamma_0$, contrary to the definition of U_0.

Therefore, one can safely assume that, for every $P \in U_0$, γ_1 and γ_2 meet transversally at P, that is,

$$f_1^{(s}(t_P) \neq f_2^{(s}(t_P), \quad \text{for some } s, \ 0 < s < r. \quad (20)$$

The results obtained in Sec. II(a)–(c) imply the existence of an open neighborhood U_0 near P_0 having the following property: Through every point P of U_0 it is possible to draw two integral curves of (1), γ_1 and γ_2, such that P is isolated in $\gamma_1 \cap \gamma_2$ and in addition γ_1 and γ_2 have a contact of order $(r-2)$ with γ_0, respectively, at P_0 and P_1.

(d) Assume now the $\mathbf{S}$ is a pointlike symmetry vector of (1) such that any integral curve of (1) having a contact of order $(r-2)$ with γ_0 either at P_0 or P_1 is invariant under the local one-parameter group of transformations generated by $\mathbf{S}$. That is, the graph $\{(t, f(t))\}$ corresponding to any solution $f(t)$ having this property will be left invariant by any member g of the local one-parameter group G generated by $\mathbf{S}$.

Under these circumstances, γ_1 and γ_2 will be invariant under G and, accordingly, the same thing will happen with $\gamma_1 \cap \gamma_2$. Now, since P is isolated in $\gamma_1 \cap \gamma_2$, P will be left invariant under the action of any $g \in G$ sufficiently close to the identity transformation, by continuity. This proves that $\mathbf{S}$ vanishes at P: since P was an arbitrary point of U_0, we conclude that $\mathbf{S}$ vanishes on U_0.

(e) Let us now compute the number of conditions sufficient in order that any integral curve of (1) having a contact of order $(r-2)$ with γ_0 at P_0 or P_1 be, as a subset of R^2, invariant under the local one-parameter group G generated by $\mathbf{S}$ (in short, under $\mathbf{S}$).

If $\mathbf{S}$ is given by

$$\mathbf{S} = \varphi(t,x)\frac{\partial}{\partial t} + \psi(t,x)\frac{\partial}{\partial x}, \quad (21)$$

then $\mathbf{S}^{(r-1)}$, the extension of $\mathbf{S}$ to the variables $t, x, \dot{x}, \ldots, x^{(r-1}$, will be given by

$$\mathbf{S}^{(r-1)} = \mathbf{S} + \sum_{i=1}^{r-1} \psi^i \frac{\partial}{\partial x^{(i}}, \quad (22)$$

where

$$\psi^j = \frac{d\psi^{j-1}}{dt} - x^{(j}\frac{d\varphi}{dt}, \qquad \psi^0 = \psi \quad \text{by definition} \quad (23)$$

and, of course,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \cdots + x^{(k}\frac{\partial}{\partial x^{(k-1}} + \cdots. \quad (24)$$

First of all, we notice that a sufficient condition in order that an integral curve of (1), $\gamma = \{(t, x(t))\}$, be invariant under $\mathbf{S}$ is that $\mathbf{S}^{(r-1)}$ vanish on its initial conditions $(t_0, x(t_0), \ldots, x^{(r-1}(t_0))$, since $\mathbf{S}$ is by hypothesis a symmetry vector of (1). Therefore, in order that $\mathbf{S}$ leave invariant *any* integral curve of (1) having a contact of order $(r-2)$ with γ_0 at P_0 or P_1 it will be sufficient that

$$\mathbf{S}^{(r-1)}\big|_{(t_0, x_0, \ldots, x_0^{(r-2}, x^{(r-1})} = 0 \quad (25a)$$

and

$$\mathbf{S}^{(r-1)}\big|_{(t_1, x_1, \ldots, x_1^{(r-2}, x^{(r-1})} = 0 \quad (25b)$$

hold for *every* value of $x^{(r-1}$.

Conditions (25a) are clearly equivalent to the following set of $(r+1)$ equalities:

$$\begin{aligned} &\varphi(t_0, x_0) = 0, \\ &\psi(t_0, x_0) = 0, \\ &\psi^1(t_0, x_0, \dot{x}_0) = 0, \\ &\cdots \\ &\psi^{r-2}(t_0, x_0, \ldots, x_0^{(r-2}) = 0, \\ &\psi^{r-1}(t_0, x_0, \ldots, x_0^{(r-2}, x^{(r-2}) = 0, \quad \forall x^{(r-1} \in \mathbb{R}, \end{aligned} \quad (26)$$

where the functions ψ^i were defined by (23). Since, for $i > 1$, the functions ψ^i are easily seen to have the following affine structure,

$$\psi^i = A_i(t, x, \ldots, x^{(i-1})x^{(i} + B_i(t, x, \ldots, x^{(i-1}), \quad (27)$$

conditions (26) are equivalent to the following set of $(r+2)$ equations:

$$\begin{aligned} &\varphi(t_0, x_0) = \psi(t_0, x_0) = \psi^i(t_0, x_0, \ldots, x_0^{(i}) = 0, \\ &i = 1, \ldots, r-2, \\ &A_{r-1}(t_0, x_0, \ldots, x_0^{(r-2}) = B_{r-1}(t_0, x_0, \ldots, x_0^{(r-2}) = 0 \end{aligned} \quad (28)$$

[notice that $r > 2$ by hypothesis, and therefore $r - 1 > 1$ implies that ψ^{r-1} has indeed the affine structure (27) with $i = r - 1$].

Conditions (25a), and hence (28), imply (as has been remarked above) that any integral curve of (1) having a contact of order $(r-2)$ with γ_0 at P_0 is invariant under $\mathbf{S}$. In particular, if (28) holds, then γ_0 itself is invariant under $\mathbf{S}$, and therefore $\mathbf{S}$ has to be parallel to the tangent vector to γ_0 on every point of γ_0, that is,

$$\mathbf{S}_P = a(t)\left[\frac{\partial}{\partial t} + \dot{\phi}(t)\frac{\partial}{\partial x}\right] \qquad \forall P = (t, \phi(t)) \in \gamma_0 \quad (29)$$

for some C^∞ function $a(t)$; by setting equal the coefficients of $\partial/\partial t$ in both members of (29), we conclude that $a(t) = \varphi(P)$ and therefore

$$\mathbf{S}_P = \varphi(P)\left[\frac{\partial}{\partial t} + \dot{\phi}(t)\frac{\partial}{\partial x}\right] \quad \forall P = (t, \phi(t)) \in \gamma_0. \quad (30)$$

Therefore, in order that P_1 be invariant under $\mathbf{S}$ a single condition suffices, namely,

$$\varphi(t_1,x_1)=0. \tag{31}$$

When this last condition holds, the invariance of γ_0 under **S** implies that any linear element of order k at P_1, $(t_1,x_1,\ldots,x_1^{(k)})$, is invariant under $\mathbf{S}^{(k)}$, for every value of k. Consequently, $\mathbf{S}^{(k)}$ vanishes at the point $(t_1,x_1,\ldots,x_1^{(k)})$ for every value of k, in particular for $k=1,\ldots,r-1$; hence we have

$$\begin{aligned}&\psi^i(t_1,x_1,\ldots,x_1^{(i)})=0,\quad i=1,\ldots,r-2,\\ &A_{r-1}(t_1,x_1,\ldots,x_1^{(r-2)})\cdot x_1^{(r-1)}+B_{r-1}(t_1,\ldots,x_1^{(r-2)})=0\end{aligned} \tag{32}$$

as a consequence of (28) and (31). Therefore, in order that (25b) be also satisfied, only one additional condition is sufficient (and not two, as it would seem), namely,

$$A_{r-1}(t_1,x_1,\ldots,x_1^{(r-2)})=0. \tag{33}$$

Indeed, using the last Eq. (32), we get

$$A_{r-1}x^{(r-1)}+B_{r-1}\overset{(32)}{=}A_{r-2}(x^{(r-1)}-x_1^{(r-1)})=0 \tag{34}$$

for every value of $x^{(r-1)}$ if and only if (33) holds [of course, A_{r-1} and B_{r-1} have to be evaluated at $(t_1,x_1,\ldots,x_1^{(r-2)})$ in (34)].

Therefore, the $(r+4)$ conditions (28), (31), and (33) are sufficient in order that any integral curve of (1) having a contact of order $(r-2)$ with γ_0 at P_0 or P_1 be invariant under the symmetry vector of (1), **S** given by (21).

(f) Let us show finally that, when $r>2$, Eq. (1) does not admit more than $r+4$ linearly independent symmetry vectors.

Indeed, suppose that $\mathbf{S}_1,\ldots,\mathbf{S}_{r+5}$ are $r+5$ symmetry vectors of (1). Since the conditions in order that a vector field be a symmetry vector of (1) constitute a system of *linear* partial differential equations, any linear combination

$$\mathbf{X}=\sum_{i=1}^{r+5}c_i\mathbf{S}_i \tag{35}$$

of $\mathbf{S}_1,\ldots,\mathbf{S}_{r+5}$ will also be a symmetry vector of (1).

On the other hand, conditions (28), (31), and (33) are easily seen to be linear in the components of the vector field **S**, by the linearity of the functions ψ^i in these components. Therefore, imposing that **X** satisfy conditions (28), (31), and (33), we obtain a linear and homogeneous system of $r+4$ algebraic equations in the unknowns $c_1,\ldots,c_{r+5}$, whose coefficients are real numbers depending on the vector fields $\mathbf{S}_1,\ldots,\mathbf{S}_{r+5}$ and on the fixed values of $(t_0,x_0,\ldots,x_0^{(r-2)})$ and $(t_1,x_1,\ldots,x_1^{(r-2)})$. Since the number of equations in this system exceeds the number of unknowns, it has a nontrivial solution $c_1^0,\ldots,c_{r+5}^0$, and, consequently, the vector field

$$\mathbf{X}_0=\sum_{i=1}^{r-5}c_i^0\mathbf{S}_i \tag{36}$$

will satisfy conditions (28), (31), and (33). Hence $\mathbf{X}_0$ must vanish on U_0, the open set defined in II(c), and, consequently,

$$\sum_{j=1}^{r+5}c_j^0\mathbf{S}_j=\mathbf{0} \tag{37}$$

on U_0, implying that $\mathbf{S}_1,\ldots,\mathbf{S}_{r-5}$ are linearly dependent on U_0, contrary to our initial assumption. This completes the proof that for $r>2$ there are at most $(r+4)$ independent symmetry vectors of (1).

(g) The case $r=2$ must be considered separately, since for $i=1$ the affine structure of ψ^i, given by

$$\psi^i=A_i(t,x,\ldots,x^{(i-1)})x^{(i}+B_i(t,x,\ldots,x^{(i-1)}) \tag{38}$$

is no longer valid, and therefore the previous reasonings fail. Indeed, we are going to see that the maximum number of independent symmetry vectors of (1) is equal to *eight* when $r=2$.

In order to prove this statement, we start from the expression of $\mathbf{S}^1$, the first extension of **S**:

$$\mathbf{S}^1=\mathbf{S}+[\psi_{,t}+(\psi_{,x}-\varphi_{,t})\dot{x}-\varphi_{,x}\cdot\dot{x}^2]\,\frac{\partial}{\partial\dot{x}}. \tag{39}$$

The line element $(t_0,x_0,\dot{x})$ will be invariant under $\mathbf{S}^1\forall\dot{x}$ provided that the following *five* conditions are satisfied:

$$\begin{aligned}&\varphi|_{P_0}=\psi|_{P_0}=0,\\ &\psi_{,t}|_{P_0}=(\psi_{,x}-\varphi_{,t})|_{P_0}=\varphi_{,x}|_{P_0}=0,\end{aligned} \tag{40}$$

where $P_0=(t_0,x_0)$ as before. Denoting again by $\phi(t)$ the solution of the differential equation

$$\ddot{x}=F(t,x,\dot{x}), \tag{41}$$

corresponding to the initial conditions $(t_0,x_0,\dot{x}_0)$, only *one* condition is now sufficient in order that a second point $P_1=(t_1,x_1)$ chosen on the integral curve of (41) associated to the solution $\phi(t)$ be invariant under the symmetry vector **S** of (41), namely,

$$\varphi(P_1)=0 \tag{42}$$

exactly as in Sec. II(e).

When (40) and (42) are satisfied, both P_1 and the integral curve γ_0 of (41) associated with the solution $\phi(t)$ are invariant under S, and, consequently, the line element $(t_1,\phi(t_1),\dot{\phi}(t_1))$ will be also invariant under S^1. The following relation is therefore automatically satisfied:

$$\psi_{,t}|_{P_1}=-(\psi_{,x}-\varphi_{,t})|_{P_1}\dot{\phi}(t_1)+\varphi_{,x}|_{P_1}\dot{\phi}^2(t_1), \tag{43}$$

leading to

$$\begin{aligned}\mathbf{S}^1|_{(t_1,x_1,\dot{x})}=\mathbf{S}|_{P_1}&+(\psi_{,x}-\varphi_{,t})|_{P_1}(\dot{x}-\dot{x}_1)\\ &+\varphi_{,x}|_{P_1}(\dot{x}_1^2-\dot{x}^2),\end{aligned} \tag{44}$$

where $\dot{x}_1=\dot{\phi}(t_1)$.

If one now imposes on $\mathbf{S}^1$ the *two* additional conditions

$$(\psi_{,x}-\varphi_{,t})|_{P_1}=0,\quad \varphi_{,x}|_{P_1}=0, \tag{45}$$

then any line element of the form $(t_1,x_1,\dot{x})$ will be left invariant by $\mathbf{S}^1\forall\dot{x}$.

Consequently, the *eight* conditions (40), (42), and (45) replace the $(r+4)$ conditions obtained when $r>2$, and, therefore, by the reasoning following in Sec. II(f), we conclude that Eq. (41) has at most *eight* independent symmetry vectors.

(h) We shall see in this section that the upper bounds on the number of independent symmetry vectors of Eq. (1) obtained above cannot be improved. Indeed, it is a standard result[5] that for $r=2$ the equation

$$\ddot{x}=0 \tag{46}$$

has exactly *eight* independent symmetry vectors; on the other hand, we are going to prove now that the equation

$$x^{(r} = 0 \tag{47}$$

has exactly $r + 4$ independent symmetry vectors when $r > 2$. Thus the upper bounds obtained above are actually attained by (47) for every $r \geqslant 2$ and therefore cannot be improved.

Let us prove that (47) has exactly $r + 4$ independent symmetry vectors when $r > 2$.

Indeed, calling $\mathbf{S}^n$ the nth extension of $\mathbf{S}$, we have

$$\mathbf{S}^n = \varphi(t,x)\frac{\partial}{\partial t} + \psi(t,x)\frac{\partial}{\partial x} + \psi^1(t,x,\dot{x})\frac{\partial}{\partial \dot{x}} + \cdots + \psi_n(t,x,\ldots,x^{(n)})\frac{\partial}{\partial x^{(n}}. \tag{48}$$

It is easy to verify that the following identity holds:

$$\psi^i = \frac{d^i\psi}{dt^i} - \sum_{k=1}^{i}\binom{i}{k}x^{(i-k+1}\frac{d^k\varphi}{dt^k}. \tag{49}$$

The condition to be satisfied in order that Eq. (1) admit $\mathbf{S}$ as a symmetry vector can be written in compact form as follows:

$$\mathbf{S}^r(x^{(r} - F) = 0 \quad \text{if } x^{(r} - F(t,x,\ldots,x^{(r-1}) = 0, \tag{50}$$

i.e., the subset of the space $\{(t,x\ldots,x^{(r})\}$ defined by

$$x^{(r} - F(t,x,\ldots,x^{(r-1}) = 0 \tag{51}$$

must be invariant under the rth extension of $\mathbf{S}$.

For the particular case of Eq. (47), condition (50) reads

$$\mathbf{S}^r(x^{(r}) = 0 \quad \text{if } x^{(r} = 0, \tag{52}$$

that is,

$$\psi^r(t,x,\ldots,x^{(r-1},0) = 0. \tag{53}$$

Taking into account the structure of ψ^i, given by (49), Eq. (53) reduces to

$$\left[\frac{d^r\psi}{dt^r} - \sum_{k=1}^{r}\binom{r}{k}x^{(r-k+1}\frac{d^k\varphi}{dt^k}\right]_{x^{(r}=0} = 0. \tag{54}$$

Let us see now that (54) has indeed $(r + 4)$ independent solutions $(\varphi(t,x),\psi(t,x))$.

In order to show this, consider first the solutions of (54) with $\varphi = 0$ given by

$$\psi(t,x) = c_1x + c_2 + c_3t + \cdots + c_{r+1}t^{r-1}, \qquad \varphi(t,x) = 0. \tag{55}$$

These particular solutions of (54) provide a set of $(r + 1)$ independent symmetry vectors of (47).

Next, since (54) is free from $\dot{\varphi}$ [the coefficient of $\dot{\varphi}$ in (54) being $x^{(r}$, which must be set equal to zero], another solution of (54) is obviously given by

$$\dot{\varphi} = a, \quad a \in R, \quad \psi = 0, \tag{56}$$

that is,

$$\varphi = at + b, \quad \psi = 0. \tag{57}$$

We have therefore $(r + 3)$ independent solutions of (54), given by (55) and (57). The additional independent solution of (54) is easily found taking into account the identity

$$\frac{d^p(tx)}{dt^p} = tx^{(p} + \binom{p}{1}x^{(p-1}, \quad p \in N, \tag{58}$$

whence we get

$$\left.\frac{d^r(tx)}{dt^r}\right|_{x^{(r}=0} = rx^{(r-1}. \tag{59}$$

Therefore, if we look for a symmetry vector having the structure

$$\mathbf{S} = tx\frac{\partial}{\partial x} + \varphi(t,x)\frac{\partial}{\partial t}, \tag{60}$$

the following relation should be satisfied by $\varphi(t,x)$:

$$rx^{(r-1} - \binom{r}{2}x^{(r-1}\ddot{\varphi} - \cdots - \binom{r}{r}\dot{x}\frac{d^r\varphi}{dt^r} = 0. \tag{61}$$

A particular solution of (61) is obviously

$$\varphi = t^2/(r-1). \tag{62}$$

Multiplying $\varphi(t,x) = t^2/(r-1)$ and $\psi(t,x) = tx$ by the factor $(r-1)$, we arrive at the following solution of (54):

$$\varphi = t^2, \quad \psi = (r-1)tx, \tag{63}$$

which is clearly independent of the other $(r + 3)$ solutions of (54) previously found, given by (55) and (57).

Therefore, (54) has at least $r + 4$ independent solutions (55), (57), and (63), and hence (47) has *at least* $r + 4$ independent symmetry vectors: since for $r > 2$ it has *at most* $r + 4$ independent symmetry vectors, as we proved in Sec. II(f), it follows that (47) has *exactly* $r + 4$ independent symmetry vectors when $r > 2$.

The reader should notice that these $(r + 4)$ symmetry vectors do behave, under the Lie–Jacobi bracket, as the generators of a Lie group. That is, one can write

$$[\mathbf{S}_i,\mathbf{S}_j] = \sum_{k=1}^{r+4}c_{ij}^k\mathbf{S}_k, \quad i,j = 1,\ldots,r+4. \tag{64}$$

This property follows from the fact that if $\mathbf{S}_i$ and $\mathbf{S}_j$ are two symmetries of (1), then the same thing will happen also with their Lie–Jacobi bracket $[\mathbf{S}_i,\mathbf{S}_j]$.

Indeed, the condition that $\mathbf{S}_k$ be a symmetry vector of (1) can be written as follows[6]:

$$[\mathbf{S}_k^{r-1},\mathbf{X}] = f_k(t,x,\ldots,x^{(r-1})\mathbf{X}, \tag{65}$$

$\mathbf{X}$ being the vector field canonically associated with Eq. (1):

$$\mathbf{X} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \cdots + F(t,x,\ldots,x^{(r-1})\frac{\partial}{\partial x^{(r-1}}. \tag{66}$$

On the other hand, we have the following identity[7]:

$$[\mathbf{A},\mathbf{B}]^P = [\mathbf{A}^P,\mathbf{B}^P], \quad p \in N, \tag{67}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary vector fields.

Therefore, since $\mathbf{S}_i$ and $\mathbf{S}_j$ are by hypothesis symmetries of (1), we have

$$\begin{aligned}
[[\mathbf{S}_i,\mathbf{S}_j]^{r-1},\mathbf{X}] &\overset{(67)}{=} [[\mathbf{S}_i^{r-1},\mathbf{S}_j^{r-1}],\mathbf{X}] \\
&= (\text{Jacobi's identity}) - [[\mathbf{S}_j^{r-1},\mathbf{X}],\mathbf{S}_i^{r-1}] \\
&\quad - [[\mathbf{X},\mathbf{S}_i^{r-1}],\mathbf{S}_j^{r-1}] \\
&\overset{(65)}{=} -[f_j\mathbf{X},\mathbf{S}_i^{r-1}] + [f_i\mathbf{X},\mathbf{S}_j^{r-1}] \\
&\overset{(65)}{=} f_j(f_i\mathbf{X}) + (\mathbf{S}_i^{r-1}f_j)\mathbf{X} \\
&\quad - f_i(f_j\mathbf{X}) - (\mathbf{S}_j^{r-1}f_i)\mathbf{X} \\
&= g_{ij}\mathbf{X},
\end{aligned} \tag{68}$$

with

$$g_{ij} = (\mathbf{S}_i^{r-1} f_j) - (\mathbf{S}_j^{r-1} f_i). \tag{69}$$

Hence the Lie–Jacobi bracket $[\mathbf{S}_i, \mathbf{S}_j]$ satisfies (65) and is therefore a symmetry vector of (1).

It easily follows that $[\mathbf{S}_i, \mathbf{S}_j]$ must be a linear combination of $\mathbf{S}_1, \ldots, \mathbf{S}_{r+4}$, since if this were not the case (47) would have $r+5$ independent symmetry vectors: $\mathbf{S}_1, \ldots, \mathbf{S}_{r+4}$ and $[\mathbf{S}_i, \mathbf{S}_j]$, contrary to what has been already proved in Sec. II(f) (since $r > 2$). Obviously, the same conclusion holds for Eq. (46).

III. MAXIMUM NUMBER OF INDEPENDENT SYMMETRY VECTORS OF THE SYSTEM $\mathbf{x}^{(r} = \mathbf{0}$ $(r > 2)$

We show in this section that a system of differential equations of the form

$$\mathbf{x}^{(r} = \mathbf{F}(t, \mathbf{x}, \ldots, \mathbf{x}^{(r-1}), \qquad \mathbf{x} = (x_1, \ldots, x_n) \in R^n, \quad \mathbf{F} \in C^\infty, \quad \text{and} \quad r > 2, \tag{70}$$

does not admit more than $2n^2 + nr + 2$ independent symmetry vectors. It would be nice to produce an example of a system of differential equations of the form (70) with $n > 1$ possessing this maximum number of independent symmetry vectors. Unfortunately, the system

$$\mathbf{x}^{(r} = \mathbf{0}, \quad r > 2, \tag{71}$$

has only $n^2 + nr + 3$ independent symmetry vectors, which is equal to the previously quoted upper bound $2n^2 + nr + 2$ only when $n = 1$. Therefore, the open problem remains of either showing that the system $\mathbf{x}^{(r} = \mathbf{0}$ has more independent symmetry vectors than any system of type (70)—in which case the number $2n^2 + nr + 2$ should be substituted by the number $n^2 + nr + 3$ as an upper bound on the number of independent symmetry vectors of (70)—or of producing a concrete example of a differential system of type (70) with the maximum number s of independent symmetry vectors $(n^2 + nr + 3 < s \leqslant 2n^2 + nr + 2)$.

(a) Let γ_0 be the integral curve of (70) corresponding to the initial conditions

$$(t_0, \mathbf{x}_0, \dot{\mathbf{x}}_0, \ldots, \mathbf{x}_0^{(r-1)}) \tag{72}$$

and $P_1 = (t_1, \mathbf{x}_1)$ be a point on γ_0 sufficiently close to $P_0 = (t_0, \mathbf{x}_0)$. By a reasoning completely similar to that followed in Secs. II(a), (b), (c), one can prove that there exists an open neighborhood $U \subset R \times R^n$ near P_0 such that through every point P of U it is possible to draw two integral curves of (70), γ_1 and γ_2, with the following two properties:

(i) γ_1 and γ_2 have a contact of order $(r-2)$ with γ_0, respectively, at P_0 and P_1.

(ii) P is isolated in $\gamma_1 \cap \gamma_2$.

(b) Assume now that the vector $\mathbf{S}$ defined by

$$\mathbf{S} = \varphi(t, \mathbf{x}) \frac{\partial}{\partial t} + \sum_{i=1}^{n} \psi_i(t, \mathbf{x}) \frac{\partial}{\partial x_i} \tag{73}$$

is a symmetry vector of Eqs. (70). If we were able to construct $\mathbf{S}$ in such a way that any integral curve of (70) having a contact of order $(r-2)$ with γ_0 either at P_0 or P_1 be invariant under the local one-parameter group G generated by $\mathbf{S}$, then in particular the two paths γ_1 and γ_2 considered above would be invariant under G, and, consequently,

$$g(\gamma_1 \cap \gamma_2) \subset \gamma_1 \cap \gamma_2 \quad \forall g \in G. \tag{74}$$

But, by construction, P is isolated in $\gamma_1 \cap \gamma_2$; therefore, we can write

$$g(P) = P \tag{75}$$

for any $g \in G$ sufficiently close to the identity transformation. Hence $\mathbf{S}$ must vanish at P, and, since P was an arbitrary point of U, we conclude that $\mathbf{S}$ is identically zero on U.

(c) Let us now show that in order that any integral curve of (70) having a contact of order $(r-2)$ with γ_0 at P_0 or P_1 be invariant under $\mathbf{S}$, $2n^2 + nr + 2$ linear conditions on $\mathbf{S}$ suffice.

First, we must impose that the linear element of order $r-1$

$$(t_0, \mathbf{x}_0, \ldots, \mathbf{x}_0^{(r-2}, \mathbf{x}^{(r-1})) \tag{76}$$

be invariant under $\mathbf{S}^{r-1}$ for every value of $\mathbf{x}^{(r-1}$, that is,

$$\mathbf{S}^{r-1}|_{(t_0, \mathbf{x}_0, \ldots, \mathbf{x}_0^{(r-2}, \mathbf{x}^{(r-1})} = \mathbf{0} \quad \forall \mathbf{x}^{(r-1} \in R^n. \tag{77}$$

Condition (77) can be written in detail as follows:

$$\begin{aligned} &\varphi(t_0, \mathbf{x}_0) = 0, \\ &\boldsymbol{\psi}(t_0, \mathbf{x}_0) = 0, \\ &\boldsymbol{\psi}^k(t_0, \mathbf{x}_0, \ldots, \mathbf{x}_0^{(k)}) = \mathbf{0}, \quad k = 1, \ldots, r-2, \\ &\boldsymbol{\psi}^{r-1}(t_0, \mathbf{x}_0, \ldots, \mathbf{x}_0^{(r-2}, \mathbf{x}^{(r-1}) = \mathbf{0}, \quad \forall \mathbf{x}^{(r-1} \in R^n, \end{aligned} \tag{78}$$

where, of course, $\boldsymbol{\psi}^k = (\psi_1^k, \ldots, \psi_n^k)$. Taking into account the identity [analogous to (49)]

$$\psi_i^{r-1} = \frac{d^{r-1}\psi_i}{dt^{r-1}} - \sum_{k=1}^{r-1} \binom{r-1}{k} x_i^{(r-k} \frac{d^k\varphi}{dt^k}, \quad i = 1, \ldots, n \tag{79}$$

and the structure of $d^k f / dt^k$, given by

$$\frac{d^k f}{dt^k} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i^{(k} + B(t, \mathbf{x}, \ldots, \mathbf{x}^{(k-1}), \qquad f: (t, \mathbf{x}) \mapsto f(t, \mathbf{x}) \in \mathbb{R}, \tag{80}$$

we conclude that ψ_i^{r-1} has the following affine structure:

$$\psi_i^{r-1} = \sum_{j=1}^{n} A_{ij}(t, \mathbf{x}, \dot{\mathbf{x}}) x_j^{(r-1} + B_i(t, \mathbf{x}, \ldots, \mathbf{x}^{(r-2}), \qquad A_{ij} = \frac{\partial \psi_i}{\partial x_j} - \dot{x}_i \frac{\partial \varphi}{\partial x_j} - (r-1) \frac{d\varphi}{dt} \delta_{ij}, \tag{81}$$

provided that $r - 1 > 1$, i.e., $r > 2$.

Therefore, taking (81) into account, (78) is equivalent to the following set of $n^2 + nr + 1$ linear conditions on the components of $\mathbf{S}$:

$$\begin{aligned} &\varphi(t_0, \mathbf{x}_0) = \psi_i(t_0, \mathbf{x}_0) = 0, \\ &\psi_i^k(t_0, \mathbf{x}_0, \ldots, \mathbf{x}_0^{(k)}) = 0, \\ &A_{ij}(t_0, \mathbf{x}_0, \dot{\mathbf{x}}_0) = 0, \\ &B_i(t_0, \mathbf{x}_0, \ldots, \mathbf{x}_0^{(r-2)}) = 0, \\ &i, j = 1, \ldots, n, \quad k = 1, \ldots, r-2. \end{aligned} \tag{82}$$

Next, in order to assure that the linear element at P_1

$$(t_1,\mathbf{x}_1,\ldots,\mathbf{x}_1^{(r-2},\mathbf{x}^{(r-1}),\quad \mathbf{x}_1^{(k} = \boldsymbol{\phi}^{(k}(t_1),\tag{83}$$

is invariant under $\mathbf{S}^{(r-1}$ for every value of $\mathbf{x}^{(r-1}$ [where $\boldsymbol{\phi}(t)$ is of course the solution of (70) corresponding to the initial conditions $(t_0,\mathbf{x}_0,\ldots,\mathbf{x}_0^{(r-1})$] we must impose that

$$\mathbf{S}^{r-1}|_{(t_1,\mathbf{x}_1,\ldots,\mathbf{x}_1^{(r-2},\mathbf{x}^{(r-1})} = \mathbf{0}\quad \forall \mathbf{x}^{(r-1}\in\mathbb{R}^n.\tag{84}$$

Now, if $\mathbf{S}$ satisfies conditions (82), then the integral curve γ_0 will be invariant under $\mathbf{S}$, since $\mathbf{S}$ is by hypothesis a symmetry vector of (70). Therefore, $\mathbf{S}$ must be parallel to the tangent vector to γ_0 on every point of γ_0, that is,

$$\mathbf{S}_P = \varphi(P)\left[\frac{\partial}{\partial t} + \sum_{i=1}^n \phi_i(t)\frac{\partial}{\partial x_i}\right],\quad \forall P=(t,\boldsymbol{\phi}(t))\in\gamma_0.\tag{85}$$

Consequently, P_1 will remain invariant under $\mathbf{S}$ if

$$\varphi(t_1,\mathbf{x}_1) = 0.\tag{86}$$

This last condition automatically implies that the linear element at P_1

$$(t_1,\mathbf{x}_1,\ldots,\mathbf{x}_1^{(k})\tag{87}$$

is invariant under $\mathbf{S}^k$ for every value of k; therefore, for $k=r-1$ we have, taking into account (81);

$$B_i(t_1,\mathbf{x}_1,\ldots,\mathbf{x}_1^{(r-2}) = -\sum_{j=1}^n A_{ij}(t_1,\mathbf{x}_1,\dot{\mathbf{x}}_1)x_1^{(r-1},\quad i=1,\ldots,n.\tag{88}$$

Consequently, the linear element (83) will be invariant under $\mathbf{S}^{r-1}$ if

$$\sum_{j=1}^n A_{ij}(t_1,\mathbf{x}_1,\dot{\mathbf{x}}_1)(x_j^{(r-1} - x_{1j}^{(r-1}) = 0,\quad i=1,\ldots,n,$$
$$\mathbf{x}^{(r-1} = (x_1^{(r-1},\ldots,x_n^{(r-1}),\tag{89}$$
$$\mathbf{x}_1^{(r-1} = (x_{11}^{(r-1},\ldots,x_{1n}^{(r-1}).$$

Since (89) must hold for every value of $\mathbf{x}^{(r-1}$, we must finally impose that

$$A_{ij}(t_1,\mathbf{x}_1,\dot{\mathbf{x}}_1) = 0,\quad i,j=1,\ldots,n.\tag{90}$$

The $2n^2+nr+2$ equations (82), (86), and (90) guarantee that any integral curve of (70) having a contact of order $(r-2)$ with γ_0 at P_0 or P_1 be invariant under the symmetry vector of (70) $\mathbf{S}$. The linearity of these equations in the components of $\mathbf{S}$ is a direct consequence of the linearity of $\mathbf{S}^{(r-1}$.

(d) We shall now compute the maximum number of independent symmetry vectors of the system

$$\mathbf{x}^{(r} = 0,\quad \mathbf{x} = (x_1,\ldots,x_n),\quad r>2.\tag{91}$$

By the reasoning given in Sec. II(h) they will automatically close as a Lie algebra under the Lie-Jacobi bracket.

The necessary and sufficient conditions in order that the vector field

$$\mathbf{S} = \varphi(t,\mathbf{x})\frac{\partial}{\partial t} + \sum_{i=1}^n \psi_i(t,\mathbf{x})\frac{\partial}{\partial x_i}$$

be a symmetry vector of (91) can be written as follows:

$$\boldsymbol{\psi}^r|_{\mathbf{x}^{(r}=\mathbf{0}} = \mathbf{0}\tag{92}$$

or, taking into a account (79),

$$\left.\frac{d^r\psi_i}{dt^r}\right|_{\mathbf{x}^{(r}=0} - \sum_{k=2}^r\binom{r}{k}x_i^{(r-k+1}\left.\frac{d^k\varphi}{dt^k}\right|_{\mathbf{x}^{(r}=0} = 0,\quad i=1,\ldots,n.\tag{93}$$

At this point it is important to have in mind the structure of $d^sf(t,\mathbf{x})/dt^s$, which can be shown to be

$$\frac{d^sf}{dt^s} = \sum_{|r|+q=s} c^s_{r_1\cdots r_p} \times \sum_{i_1,\ldots,i_p=1}^n \frac{\partial^{p+q}f}{\partial t^q\,\partial x_{i_1}\cdots\partial x_{i_p}}x_{i_1}^{(r_1}\cdots x_{i_p}^{(r_p},$$
$$|r| = r_1+\cdots+r_p,\quad r_1<r_2<\cdots<r_p,\quad c^s_{r_1\cdots r_p}\in N.\tag{94}$$

An immediate consequence of (94) is that a term of the form $x_i^{(r-1}\ddot{x}_j$ cannot appear in (93) from the development of $d^r\psi_i/dt^{r-1}$, since $(r+1)+2 = r+1>r$. A term of this type can only arise, therefore, from the expressions

$$-\binom{r}{2}x_i^{(r-1}\frac{d^2\varphi}{dt^2},\quad -\binom{r}{r-1}\frac{d^{r-1}\varphi}{dt^{r-1}}\tag{95}$$

also appearing in (93). These two terms are different when $r-1\neq 2$, i.e., when $r>3$, and therefore for $r>3$ the coefficient of the term $x_i^{(r-1}\ddot{x}_j$ is either

$$-\binom{r}{2}\varphi_{,j},\quad \text{when } i\neq j,$$

or (96)

$$-\left[\binom{r}{2}+r\right]\varphi_{,j},\quad \text{when } i=j,$$

whereas, for $r=3$, $x_i^{(r-1}\ddot{x}_j$ reduces to $\ddot{x}_i\ddot{x}_j$, whose coefficient is simply

$$-3\varphi_{,j},\quad i,j=1,\ldots,n.\tag{97}$$

Since (93) must be an *identity in* $\dot{\mathbf{x}},\ddot{\mathbf{x}},\ldots,\mathbf{x}^{(r-1}$, and φ, ψ_i do not depend on these variables, the coefficient of the term $x_i^{(r-1}\ddot{x}_j$ must equal zero; taking into account (96) (for $r>3$) and (97) (for $r=3$), we conclude that

$$\varphi_{,j} = 0,\quad j=1,\ldots,n.\tag{98}$$

Accordingly, for every symmetry vector of (91) we have

$$\varphi(t,x) = f(t).\tag{99}$$

Note that the above reasoning obviously fails when $r=2$, since then the term $x_i^{(r-1}\ddot{x}_j$ reduces to $\dot{x}_i\ddot{x}_j$, which is absent from (93) by the restriction $\ddot{\mathbf{x}}=0$.

Substituting (99) into (93), we obtain

$$\left.\frac{d^r\psi_i}{dt^r}\right|_{\mathbf{x}^{(r}=0} - \sum_{k=2}^r\binom{r}{k}x_i^{(r-k+1}f^{(k}(t) = 0,\quad i=1,\ldots,n.\tag{100}$$

Remembering (94) again, we realize that the term $x_j^{(r-1}\dot{x}_k$ appears in (100) only through $d^r\psi_i/dt^r|_{\mathbf{x}^{(r}=\mathbf{0}}$ and its coefficient is (up to the positive integer $c^r_{1,r-1}$) $\psi_{i,jk}$. Therefore, we must have

$$\frac{\partial^2\psi_i}{\partial x_j\,\partial x_k} = 0,\quad i,j,k=1,\ldots,n.\tag{101}$$

Similarly, considering the coefficients of the terms $x_j^{(r-1}$ with $j\neq 1$, which again only appear in (100) through $d^r\psi_i/dt^r|_{\mathbf{x}^{(r}=\mathbf{0}}$, we obtain

$$\frac{\partial^2\psi_i}{\partial t\,\partial x_j}=0,\quad i,j=1,...,n,\ i\neq j. \tag{102}$$

When $i=j$, considering the coefficient of the term $x_i^{(r-1}$ in (100), we get

$$c_{r-1}^r\frac{\partial^2\psi_i}{\partial t\,\partial x_i}-\binom{r}{2}\ddot f(t)=0,\quad i=1,...,n. \tag{103}$$

Since c_{r-1}^r is a positive integer, we can rewrite (103) as follows:

$$\frac{\partial^2\psi_i}{\partial t\,\partial x_i}=K\cdot\dot f(t),\quad i=1,...,n\quad \left[K=\binom{r}{2}\quad c_{r-1}^r>0\right]. \tag{104}$$

Considering now the coefficient of the term independent of $\dot{\mathbf{x}},\ddot{\mathbf{x}},...,\mathbf{x}^{(r-1}$ in (100), we are led to

$$\frac{\partial^r\psi_i}{\partial t^r}=0,\quad i=1,...,n. \tag{105}$$

From Eq. (101) we readily obtain

$$\psi_i=\sum_{j=1}^n a_{ij}(t)x_j+b_i(t), \tag{106}$$

and, taking (102) and (104) into account, we immediately arrive at

$$\psi_i=\sum_{j=1}^n a_{ij}x_j+K\dot f(t)x_i+b_i(t),$$
$$a_{ij}\in R\quad \forall i,j=1,...,n, \tag{107}$$

and, substituting (107) into (105), we finally get

$$Kf^{(r+1}(t)x_i+b_i^{(r}(t)=0. \tag{108}$$

Therefore, we must have

$$f(t)=P_r(t),\quad b_i(t)=Q_{r-1}^i(t),\quad i=1,...,n, \tag{109}$$

P_r and Q_{r-1}^i being polynomials of maximum degree r and $(r-1)$, respectively. From (107) and (109) we get the following structure of ψ_i,

$$\psi_i=\sum_{j=1}^n a_{ij}x_j+Kx_i\dot P_r(t)+Q_{r-1}^i(t)\quad (a_{ij}\in R), \tag{110}$$

and, substituting it back into (100), we arrive at

$$K\frac{d^r}{dt^r}[x_i\dot P_r(t)]\Big|_{\mathbf{x}^{(r}=\mathbf{0}}-\sum_{k=2}^r\binom{r}{k}x_i^{(r-k+1}P_r^{(k}(t)=0. \tag{111}$$

Applying Leibnitz's theorem to the first term of (111), we obtain

$$K\sum_{k=1}^{r-1}\binom{r}{k}x_i^{(r-k}P_r^{(k+1}(t)=\sum_{k=1}^{r-1}\binom{r}{k+1}x_i^{(r-k}P_r^{(k+1}(t). \tag{112}$$

Since we are considering now the case $r>2$, we can compare the coefficients of $x_i^{(r-1}$ and $x_i^{(r-2}$ in both members of (112), obtaining

$$K\cdot r\cdot\ddot P_r(t)=\binom{r}{2}\cdot\ddot P_r(t),$$
$$K\cdot\binom{r}{2}\cdot\dddot P_r(t)=\binom{r}{3}\cdot\dddot P_r(t). \tag{113}$$

It is easy to prove by induction that $c_{r-1}^r=r$; hence $K=(r-1)/2$ [see (104)]. The first equation in (113) reduces to an identity and the second one leads to

$$\dddot P_r(t)=0,\quad \forall t\in R, \tag{114}$$

i.e., $P_r(t)=a+bt+ct^2\quad (a,b,c\in R)$.

Conversely, if (114) holds, then (112) is automatically satisfied. Therefore, the "general solution" of (93) is obtained by substituting $P_r(t)=a+bt+ct^2$ into (110), and, consequently, the general solution of (92) is

$$\varphi=a+bt+ct^2,$$
$$\psi_i=\sum_{j=1}^n A_{ij}x_j+c(r-1)tx_i+Q_{r-1}^i(t), \tag{115}$$
$$i=1,...,n,\quad a,b,c,A_{ij}\in R,$$

where we have set $A_{ij}=a_{ij}+\frac{1}{2}b(r-1)\delta_{ij}$ (δ_{ij} being, of course, the Kronecker delta).

From (115) we immediately obtain the following set of n^2+nr+3 independent symmetry vectors of (91):

$$x_i\frac{\partial}{\partial x_j},\quad i,j=1,...,n,$$
$$t^p\frac{\partial}{\partial x_i},\quad p=0,1,...,r-1,\ i=1,...,n, \tag{116}$$
$$\frac{\partial}{\partial t},\quad t\frac{\partial}{\partial t},\quad t^2\frac{\partial}{\partial t}+(r-1)t\cdot\sum_{k=1}^n x_k\frac{\partial}{\partial x_k}.$$

This establishes the point we wanted to make: when $n>1$ and $r>2$, the system of differential equations $\mathbf{x}^{(r}=\mathbf{0}$ does not provide us (as happened for $n=1$) with a maximum number of independent symmetry vectors equal to the upper bound $2n^2+nr+2$ obtained in III(a)–(c). Therefore, it remains an open problem to find systems of differential equations—if any—whose maximum number of independent symmetry vectors is greater than the number n^2+nr+3.

Finally, note that, when $n=1$, the symmetry vectors (116) reduce to the symmetry vectors of $x^{(r}=0$ computed in Sec. II(h), as it should be.

IV. MAXIMUM NUMBER OF INDEPENDENT SYMMETRY VECTORS OF THE SYSTEM $\ddot{\mathbf{x}}=\mathbf{F}$

We show in this section that a system of differential equations of the form

$$\ddot{\mathbf{x}}=\mathbf{F}(t,\mathbf{x},\dot{\mathbf{x}}),\quad \mathbf{x}=(x_1,...,x_n) \tag{117}$$

cannot possess more than $2(n+1)^2$ independent symmetry vectors. We also compute, by a direct procedure, all the symmetry vectors of the system $\ddot{\mathbf{x}}=\mathbf{0}$, obtaining only n^2+4n+3 independent vectors. Since this number is less than the upper bound $2(n+1)^2$ mentioned above, the open question arises of whether or not there exist differential systems admitting more than n^2+4n+3 independent symmetry vectors.

In Sec. V we show that this is not the case: In other words, the maximum number of independent symmetry vectors admitted by any system of the form (117) is never greater than n^2+4n+3, the number of independent symmetry vectors of the system $\ddot{\mathbf{x}}=\mathbf{0}$.

(a) Let

$$S=\varphi\frac{\partial}{\partial t}+\sum_{i=1}^{n}\psi_i\frac{\partial}{\partial x_i}\tag{118}$$

be a pointlike symmetry vector of (117); then it is easy to verify that the structure of $\mathbf{S}^1$ (the first extension of $\mathbf{S}$ to the variables $t,\mathbf{x},\dot{\mathbf{x}}$) is the following:

$$\begin{aligned}\mathbf{S}^1&=\mathbf{S}+\sum_{i=1}^{n}\psi_i^1\cdot\frac{\partial}{\partial x_i},\\ \psi_i^1&=\sum_{j=1}^{n}\psi_{i,j}\dot{x}_j+\psi_{i,t}\\&\quad-\dot{x}_i\left(\sum_{j=1}^{n}\varphi_{,j}\dot{x}_j+\varphi_{,t}\right).\end{aligned}\tag{119}$$

Therefore, it is clear that the linear element $(t_0,\mathbf{x}_0,\dot{\mathbf{x}})=(P_0,\dot{\mathbf{x}})$ will be left invariant by $\mathbf{S}^1$ if for every value of $\dot{\mathbf{x}}$ the following set of n^2+3n+1 linear equations in the components of $\mathbf{S}$ holds:

$$\left.\begin{array}{l}\varphi(P_0)=0,\quad \boldsymbol{\psi}(P_0)=0\\ \psi_{i,t}(P_0)=\varphi_{,j}(P_0)=0\\ (\psi_{i,j}-\varphi_{,t}\delta_{ij})(P_0)=0\end{array}\right\},\quad i,j=1,\dots,n.\tag{120}$$

Similarly, a second point $P_1=(t_1,\mathbf{x}_1)$ lying on the integral curve γ_0 of (117) corresponding to the initial conditions $(t_0,\mathbf{x}_0,\dot{\mathbf{x}}_0)$ will be left invariant by $\mathbf{S}$ provided only that

$$\varphi(P_1)=0\tag{121}$$

since, exactly as in Secs. II and III, (121) and the fact that $\mathbf{S}$ is a symmetry vector of (117) and P_1 lies on an integral curve of (117) imply that $\boldsymbol{\psi}(P_1)=\mathbf{0}$ as well.

Finally, from all that has been said in Secs. II and III, it should be clear by now that, in order that any linear element at P_1, $(P_1,\dot{\mathbf{x}})$, be invariant under $\mathbf{S}^1$, the following n^2+n linear conditions in φ and $\boldsymbol{\psi}$ suffice:

$$\left.\begin{array}{l}\varphi_{,j}(P_1)=0\\ (\psi_{i,j}-\varphi_{,t}\delta_{ij})(P_1)=0\end{array}\right\},\quad i,j=1,\dots,n,\tag{122}$$

since when (120), (121) and (122) hold $\psi_{i,t}(P_1)$ automatically vanishes, due to the fact that the linear element $(P_1,\dot{\mathbf{x}}_1)$ tangent to γ_0 is then invariant under $\mathbf{S}^1$.

Accordingly, the $2(n+1)^2$ conditions (120), (121), and (122) are sufficient in order that any linear element at P_0 or P_1 be invariant under $\mathbf{S}^1$; since these conditions are linear in the components of $\mathbf{S}$, the same construction followed in Secs. II and III can be repeated now, with the result that Eq. (117) does not admit more than $2(n+1)^2$ independent symmetry vectors.

(b) We now compute all the pointlike symmetry vectors of the system

$$\ddot{\mathbf{x}}=\mathbf{0},\quad \mathbf{x}=(x_1,\dots,x_n)\tag{123}$$

in order to establish whether or not the dimension of the vector space generated by these symmetries equals the upper bound $2(n+1)^2$ obtained above.

Since the necessary and sufficient conditions in order that (118) be a symmetry vector of (123) are

$$\psi_i^2|_{\ddot{\mathbf{x}}=\mathbf{0}}=0,\quad i=1,\dots,n,\tag{124}$$

computing $\psi_i^2|_{\ddot{\mathbf{x}}=\mathbf{0}}$ and setting equal to zero the coefficients of 1, $\dot{x}_i$, and $\dot{x}_i\dot{x}_j$, we arrive at the following system of partial differential equations in φ, $\boldsymbol{\psi}$:

$$\left.\begin{array}{ll}\varphi_{,jk}=0 & (125\text{a})\\ \psi_{i,tt}=0 & (125\text{b})\\ \psi_{i,jk}=\varphi_{,kt}\delta_{ij}(1+\delta_{jk}) & (125\text{c})\\ \psi_{i,jt}=\tfrac{1}{2}\varphi_{,tt}\delta_{ij} & (125\text{d})\end{array}\right\},\quad i,j,k=1,\dots,n,$$

From (125a) and (125b) we get

$$\begin{aligned}\varphi&=\sum_{j=1}^{n}C_j(t)x_j+D(t),\\ \psi_i&=A_i(x)t+B_i(x).\end{aligned}\tag{126}$$

Substituting (126) into (125c) and (125d), we obtain

$$\begin{aligned}A_i(x)&=a_i(x_i),\\ B_i(x)&=b_i(x_i)+\sum_{\substack{j=1\\(j\neq 1)}}^{n}b_{ij}(x_i)x_j.\end{aligned}\tag{127}$$

Substituting (127) back into (126), we obtain, after some easy calculations, the general solution of (125):

$$\begin{aligned}\varphi&=\sum_{j=1}^{n}(c_jt+c_j')x_j+at^2+dt+d',\\ \psi_i&=(ax_i+a_i)t+\sum_{j=1}^{n}c_jx_ix_j\\&\quad+\sum_{j=1}^{n}b_{ij}x_j+b_i.\end{aligned}\tag{128}$$

From (128) we obtain the following set of n^2+4n+3 independent generators of the vector space of the symmetries of (123):

$$\left.\begin{array}{l}\dfrac{\partial}{\partial t},\quad t\dfrac{\partial}{\partial t},\quad x_i\dfrac{\partial}{\partial t}\\[2mm] \dfrac{\partial}{\partial x_i},\quad t\dfrac{\partial}{\partial x_i},\quad x_j\dfrac{\partial}{\partial x_i}\\[2mm] tx_i\dfrac{\partial}{\partial t}+x_i\displaystyle\sum_{k=1}^{n}x_k\dfrac{\partial}{\partial x_k}\\[2mm] t^2\dfrac{\partial}{\partial t}+t\displaystyle\sum_{k=1}^{n}x_k\dfrac{\partial}{\partial x_k}\end{array}\right\},\quad i,j=1,\dots,n.\tag{129}$$

By the reasoning followed in Sec. II, the set of vectors (129) closes as a Lie algebra under the Lie–Jacobi bracket.

It is not difficult to verify that the set of symmetry vectors given by (129) is a set of generators for the projective pseudogroup of the space $\{(\mathrm{t},\mathbf{x})\}=R^{n+1}$, whose finite expression is given by

$$\begin{aligned}x_i'&=\frac{\sum_{j=1}^{n+1}a_{ij}x_j+a_{i,n+2}}{\sum_{j=1}^{n+1}b_jx_j+b_{n+2}},\\ &\qquad x_{n+1}=t,\quad i=1,\dots,n+1.\end{aligned}\tag{130}$$

The projective pseudogroup does precisely possess $(n+2)^2-1=n^2+4n+3$ essential parameters, and, therefore, n^2+4n+3 independent generators (see the Appendix).

V. REDUCTION OF THE MAXIMUM NUMBER OF INDEPENDENT SYMMETRIES OF THE SYSTEM $\ddot{\mathbf{x}} = \mathbf{F}$

We show in this section that the system (117) does not admit more than $n^2 + 4n + 3$ independent symmetry vectors, thereby achieving an improvement of the maximum number of independent symmetry vectors of (117), $2n^2 + 4n + 2$, derived in Sec. IV. The new upper bound obtained in this section cannot be further improved, since in Sec. IV, it has been shown that the system $\ddot{\mathbf{x}} = \mathbf{0}$ has precisely $n^2 + 4n + 3$ independent symmetry vectors.

The proof given here uses the following remarkable property of the projective pseudogroup of R^{n+1}:

If a projective transformation T of R^{n+1} leaves $n+3$ points of R^{n+1} fixed, and these points are in "*generic position*," then T is the identity transformation.[8] (We say that $n+3$ points of R^{n+1} are in *generic position* if for *every* selection of $n+2$ of them the $n+1$ vectors obtained choosing one of these $n+2$ points as the origin and the rest as end points are linearly independent.)

(a) Let $P_1,...,P_{n+3}$ be $n+3$ points of R^{n+1} such that

$$P_i = (t_i,\mathbf{x}_i), \quad i \neq j \Rightarrow t_i \neq t_j. \tag{131}$$

Let us assume for the moment that these points can be chosen in such a way that to any couple of them (P_i,P_j) with $i \neq j$ there corresponds an integral curve $\gamma_{ij} = \{(t,\boldsymbol{\phi}_{ij}(t)) \times | t \in R\}$ of (117) passing through P_i and P_j: We shall prove in Sec. V (e) that this assumption can indeed be satisfied.

Assuming then that we have chosen the points $P_1,...,P_{n+3}$ in a such a way that this last assumption holds true, by a straightforward generalization of the argument given in Sec. II(a) one can prove the following result:

If the points $P_1,...,P_{n+3}$ are sufficiently close to each other, then for every pair (i, j) with $i \neq j$ there exist open neighborhoods U_{ij} and P_j and V_{ij} of $\dot{\mathbf{x}}_{ij} = \dot{\boldsymbol{\phi}}_{ij}(t_i)$ such that through every point P of U_{ij} there passes exactly one integral curve of (117) containing P_i, with velocity (= derivative with respect to time t) at t_i lying in V_{ij}.

Suppose now that the vector field $\mathbf{S}$ given by (118) is a symmetry of (117) leaving all the points $P_1,...,P_{n+3}$ invariant. It is clear that in order to achieve it the following $(n+1)(n+3)$ conditions are sufficient:

$$\varphi(P_i) = 0, \quad \psi_j(P_i) = 0, \quad i = 1,...,n+3, j = 1,...,n. \tag{132}$$

Equations (132) automatically imply that the integral curves γ_{ij} $(i \neq j)$ are subsets of R^{n+1} invariant under $\mathbf{S}$, and therefore that the $n+2$ linear elements at P_i

$$(P_i,\dot{\mathbf{x}}_{ij}), \quad i \neq j, \tag{133}$$

are also invariant under $\mathbf{S}^1$, for every $i = 1,...,n+3$.

Indeed, if g is a transformation belonging to the local one-parameter group generated by S and sufficiently close to the identity, then we have by continuity

$$\dot{\mathbf{x}}'_{ij} = \frac{d}{ds}\bigg|_{s=t_i} (g\boldsymbol{\phi}_{ij})(s) \in V_{ij}$$
$$[\, g\gamma_{ij} = \{(s,(g\boldsymbol{\phi}_{ij})(s)) | s \in R\}\,] \tag{134}$$

since $\dot{\mathbf{x}}_{ij} \in V_{ij}$ by construction. But this necessarily implies that $g\gamma_{ij} = \gamma_{ij}$, since both γ_{ij} and its transform $g\gamma_{ij}$ pass through P_i with velocity at P_i lying in V_{ij}, and the equality of γ_{ij} and its transform implies obviously that $\dot{\mathbf{x}}_{ij}$ equals $\dot{\mathbf{x}}'_{ij}$, its transform under S^1, as claimed.

(b) Consider now a finite transformation

$$t' = g(t,\mathbf{x}), \quad \mathbf{x}' = \mathbf{f}(t,\mathbf{x}) \tag{135}$$

such that P_i is invariant under (135), for fixed $i \in \{1,...,n+3\}$. As is well known, the transformation induced by (135) on the derivatives $\dot{\mathbf{x}}$ at P_i is given by

$$\dot{\mathbf{x}}' = \frac{\Sigma_{j=1}^n \mathbf{f}_{,j}(P_i)\dot{x}_j + \mathbf{f}_{,t}(P_i)}{\Sigma_{j=1}^n g_{,j}(P_i)\dot{x}_j + g_{,t}(P_i)}, \tag{136}$$

i.e., any curve $\{(t,a(t)) | t \in R\}$ passing through P_i such that $\dot{\mathbf{a}}(t_i) = \dot{\mathbf{x}}$ will be transformed under (135) into another curve $\{(s,\mathbf{b}(s)) | s \in \mathbb{R}\}$ through P_i, with $\dot{\mathbf{b}}(t_i) = \dot{\mathbf{x}}'$.

Since P_i is fixed, (136) implies that the velocities at P_i transform under a projective transformation, whose parameters depend, of course, on the point P_i that is being kept fixed. Denoting now by

$$t' = g(t,\mathbf{x};\alpha), \quad \mathbf{x}' = \mathbf{f}(t,\mathbf{x};\alpha), \tag{137}$$

the local one-parameter group of transformations generated by the symmetry $\mathbf{S}$ satisfying conditions (132), then $\mathbf{S}^1$ acts on the velocities at P_i as a one-parameter subgroup G_i of the projective pseudogroup of $R^n = \{\dot{\mathbf{x}}\}$. Furthermore, every transformation $g \in G_i$ leaves invariant the $n+2$ linear elements at P_i given by (133), as we have just seen: therefore, if we are able to choose the velocities $\dot{\mathbf{x}}_{ij}$ ($i \neq j$, i fixed) in generic position (by an appropriate selection of the points $P_1,...,P_{n+3}$), then, by the property of the projective pseudogroup quoted at the beginning of this section, we can conclude that G_i reduces to the identity transformation and, therefore, that $\mathbf{S}^1$ leaves *every* linear element at P_i invariant.

(c) Suppose now that we are able to find a set of $n+3$ points of R^{n+1} $\{P_1,...,P_{n+3}\}$ satisfying (131), and the following additional requirement: The *two* sets of $(n+2)$ vectors of R^n given by

$$\{\dot{\mathbf{x}}_{1j} | j = 2,3,...,n+3\}, \quad \{\dot{\mathbf{x}}_{2k} | k = 1,3,4,...,n+3\} \tag{138}$$

are in generic position in R^n. According to III(a), we can find an open neighborhood U in R^{n+1} such that through every point P of U there pass two integral curves of (117), γ_1 and γ_2, joining P, respectively, with P_1 and P_2 in such a way that P is isolated in $\gamma_1 \cap \gamma_2$. Since (131), (132), and (138) imply that every linear element at P_1 or P_2 is invariant under $\mathbf{S}$ and $\mathbf{S}$ is by hypothesis a symmetry vector of (117), it follows that γ_1 and γ_2 are both invariant under $\mathbf{S}$; therefore, P has to be invariant under $\mathbf{S}$, since P is isolated in $\gamma_1 \cap \gamma_2$. Hence every point of U is invariant under $\mathbf{S}$, implying that $\mathbf{S} = \mathbf{0}$ on U.

Since conditions (132) are clearly linear in the components of $\mathbf{S}$, we can again apply the argument of Sec. II(f) to conclude that (117) does not admit more than $n^2 + 4n + 3$ independent symmetry vectors.

The only point meriting a separate treatment in order that our proof be complete is the following: We have to show that it is indeed possible to find a set of $n+3$ points of R^{n+1} satisfying conditions (131) and (138), such that every pair of points of this set can be joined by an integral curve of (117). In order to prove this statement, the lemma that follows is of great practical value since, as we shall explain below, it reduces the problem of finding the set of points $P_1,...,P_{n+3}$ with the properties mentioned above to an easier one.

(d) *Lemma*: Let $P_0=(t_0,\mathbf{x}_0)$ be a point of R^{1+n} and call $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ the unique solution of (117) corresponding to the initial condition $(P_0,\boldsymbol{\xi})$. Consider the straight line of R^{1+n} parallel to $(1,\mathbf{v})$ and passing through P_0, whose equation is

$$t=t_0+s,\quad \mathbf{x}=\mathbf{x}_0+s\mathbf{v}\quad \forall s\in R. \tag{139}$$

Then one can find $\epsilon>0$ such that for every s such that $0<|s|<\epsilon$ there is an integral curve of (117) passing through P_0 and $(t_0+s,\mathbf{x}_0+sv)$, whose derivative at t_0, $\mathbf{h}(s)$, satisfies

$$\lim_{s\to 0}\mathbf{h}(s)=\mathbf{v}. \tag{140}$$

Proof: The function $\mathbf{f}(s,\boldsymbol{\xi})$ defined by

$$\mathbf{f}(s,\boldsymbol{\xi})=\boldsymbol{\phi}(t_0+s,\boldsymbol{\xi})-\mathbf{x}_0-s\boldsymbol{\xi} \tag{141}$$

is a C^∞ function [since the function $\mathbf{F}$ appearing in (117) is assumed in what follows to be of class C^∞][9] and satisfies

$$\mathbf{f}(0,\boldsymbol{\xi})=\mathbf{0}\quad \forall\boldsymbol{\xi}\in R^n. \tag{142}$$

Therefore, we can write

$$\mathbf{f}(s,\boldsymbol{\xi})=\int_0^1\frac{\partial}{\partial\theta}\mathbf{f}(\theta s,\boldsymbol{\xi})\,d\theta, \tag{143}$$

and, since

$$\frac{\partial}{\partial\theta}\mathbf{f}(\theta s,\boldsymbol{\xi})=s\cdot\mathbf{f}_{,s}(\theta s,\boldsymbol{\xi}), \tag{144}$$

$\mathbf{f}(s,\boldsymbol{\xi})$ can be factorized as follows:

$$\mathbf{f}(s,\boldsymbol{\xi})=s\cdot\mathbf{g}(s,\boldsymbol{\xi}),$$
$$\left(g(s,\boldsymbol{\xi})=\int_0^1 f_{,s}(\theta s,\boldsymbol{\xi})\,d\theta\right), \tag{145}$$

where $\mathbf{g}(s,\boldsymbol{\xi})$ is C^∞ since $\mathbf{f}$ is C^∞.

Therefore, we have

$$\boldsymbol{\phi}(t_0+s,\boldsymbol{\xi})=s\cdot\mathbf{g}(s,\boldsymbol{\xi})+x_0+s\boldsymbol{\xi},\quad \mathbf{g}\in C^\infty \tag{146}$$

and the intersection of the integral curve $\{(t,\boldsymbol{\phi},(t,\boldsymbol{\xi}))|t\in R\}$ with the straight line (139) leads to the equation

$$s\cdot\mathbf{g}(s,\boldsymbol{\xi})+\mathbf{x}_0+s\boldsymbol{\xi}=\mathbf{x}_0+s\mathbf{v} \tag{147a}$$

or, since $s\neq 0$,

$$\mathbf{v}=\boldsymbol{\xi}+\mathbf{g}(s,\boldsymbol{\xi}). \tag{147b}$$

Equation (144) implicitly defines $\boldsymbol{\xi}$ as a C^∞ function of s, $\boldsymbol{\xi}=\mathbf{h}(s)$. Indeed, define a function $\boldsymbol{\psi}$, $(s,\boldsymbol{\xi})$ as follows:

$$\boldsymbol{\psi}(s,\boldsymbol{\xi})=\boldsymbol{\xi}+\mathbf{g}(s,\boldsymbol{\xi})-\mathbf{v}. \tag{148}$$

Then we have

$$\boldsymbol{\psi}(0,\mathbf{v})=\mathbf{g}(0,\mathbf{v})=\mathbf{0} \tag{149}$$

[since $\mathbf{g}(0,\boldsymbol{\xi})=\int_0^1\mathbf{f}_{,s}(0,\boldsymbol{\xi})\,d\theta=\mathbf{0}\ \forall\boldsymbol{\xi}\in\mathbb{R}$ on account of the definition (141)], and

$$(D_{\boldsymbol{\xi}}\boldsymbol{\psi})(0,\mathbf{v})=I+(D_{\boldsymbol{\xi}}\mathbf{g})(0,\mathbf{v})=I$$
$$(I=\text{identity matrix of dimension } n) \tag{150}$$

[taking into account that $\mathbf{g}(0,\boldsymbol{\xi})=\mathbf{0}$ for every $\boldsymbol{\xi}$, as we have just shown].

Equations (149) and (150) allow us to apply the implicit function theorem to the function $\boldsymbol{\psi}(s,\boldsymbol{\xi})$ at the point $(0,\mathbf{v})$, thus obtaining $\boldsymbol{\xi}$ as a function of s, $\boldsymbol{\xi}=\mathbf{h}(s)$, in a sufficiently small neighborhood $|s|<\epsilon$ of $s=0$. The function $\mathbf{h}(s)$ satisfies

$$\mathbf{h}(0)=\mathbf{v}, \tag{151a}$$
$$\boldsymbol{\psi}(s,\mathbf{h}(s))=\mathbf{0}\quad \text{if } |s|<\epsilon. \tag{151b}$$

It follows that the integral curve of (117) corresponding to the initial condition $(P_0,\mathbf{h}(s))$ passes through P_0 and through the point $(t_0+s,\mathbf{x}_0+s\mathbf{v})$ [by (146)–(148) and (151)]; in addition, we have

$$\lim_{s\to 0}\mathbf{h}(s)=\mathbf{h}(0)=\mathbf{v} \tag{152}$$

on account of (151a) since $\mathbf{h}(s)$ is a continuous function (as a matter of fact, $\mathbf{h}$ is C^∞, as follows from the fact that $\mathbf{g}$ is C^∞ and the implicit function theorem). This completes the proof of the lemma.

(e) *Consequences of the lemma*: Let $\{P_1,...,P_{n+3}\}$ be a set of $n+3$ points of R^{1+n}, $P_i=(t_i,\mathbf{x}_i)$, satisfying the following conditions:

$$i\neq j,\quad t_i\neq t_j; \tag{153a}$$

the two sets of points of R^n

$$\left\{\frac{\mathbf{x}_i-\mathbf{x}_1}{t_i-t_1}\middle|\, i=2,3,...,n+3\right\},$$
$$\left\{\frac{\mathbf{x}_j-\mathbf{x}_2}{t_j-t_2}\middle|\, j=1,3,...,n+3\right\} \tag{153b}$$

are in generic position in R^n.

We shall indicate at the end of this section how to construct sets of $n+3$ points of R^{1+n} satisfying conditions (153).

Consider now the transformation $H_a:R^{1+n}\to R^{1+n}$ defined as follows:

$$H_a(P)=P_1+a(P-P_1)\equiv P^a,\quad a\in R,\ a>0. \tag{154}$$

If $\{P_1,...,P_{n+3}\}$ satisfy conditions (153), the same will happen with $\{P_1^a,...,P_{n+3}^a\}$, since we have

$$\frac{\mathbf{x}_i^a-\mathbf{x}_k^a}{t_i^a-t_k^a}=\frac{a(\mathbf{x}_i-\mathbf{x}_k)}{a(t_i-t_k)}=\frac{\mathbf{x}_i-\mathbf{x}_k}{t_i-t_k}. \tag{152}$$

When $a\to 0$, $P_i^a\to P_1^a$ for every $i=1,...,n+3$, but the directions $(t_i-t_j,\mathbf{x}_i-\mathbf{x}_j)$ defined by every pair of points P_i,P_j with $i\neq j$ remain invariant under H_a.

Therefore, by repeated application of the lemma proved above, it follows that, for sufficiently small a, for every pair of points P_i,P_j with $i\neq j$ there is an integral curve of (117) joining P_i with P_j and satisfying

$$\lim_{a\to 0}\dot{\boldsymbol{\phi}}_{ij}^a(t_i)=\frac{\mathbf{x}_j-\mathbf{x}_i}{t_j-t_i},$$
$$i,j=1,...,n+3,\quad i\neq j, \tag{156}$$

where $\boldsymbol{\phi}_{ij}^a(t)$ is the solution of (117) whose associated integral curve passes through P_i and P_j.

Furthermore, it is easy to verify that if $m+2$ points of R^m are in generic position, any sufficiently small perturbation applied to them will lead again to a set of $m+2$ points in generic position; this is essentially due to the fact that genericity is defined in terms of linear independence of certain sets of vectors, and linear independence is preserved by sufficiently small perturbations. It follows [by (153b)] that the two sets of vectors of R^n defined by

$$\{\dot{\boldsymbol\phi}^a_{1i}\,|\,i=2,\dots,n+3\},$$
$$\{\dot{\boldsymbol\phi}^a_{2j}\,|\,j=1,3,\dots,n+3\} \tag{157}$$

are in generic position in R^n, if we choose a sufficiently small.

The conclusion is, therefore, that if $P_1,\dots,P_{n+3}$ satisfy condition (153), then one can find $a\in\mathbb{R}$ such that the new set of points $P^a_1,\dots,P^a_{n+3}$ satisfy conditions (131) and (138). The only point that remains to be proved is, therefore, that it is indeed possible to find $P_1,\dots,P_{n+3}$ such that conditions (153) are satisfied.

To this end, notice that if the following points of R^n

$$\{0,\mathbf{v}_1,\dots,\mathbf{v}_n\} \tag{158}$$

are in generic position, it immediately follows that the following set of $n+3$ points of R^{1+n},

$$\{(0,\mathbf{0}),(r_0,\mathbf{0}),(r_1,\mathbf{v}_1),\dots,(r_{n+1},\mathbf{v}_{n+1})\},$$
$$r_i\neq 0\ \ \forall i=0,1,\dots,n+1,\quad r_i\neq r_j\ \ \forall i\neq j, \tag{159}$$

satisfies conditions (153), provided only that the numbers

$$r_0,\quad r_i-1,\quad i=1,\dots,n+1, \tag{160}$$

are chosen sufficiently small.

Indeed, choosing $P_1=(0,\mathbf{0})$ and $P_2=(r_0,\mathbf{0})$, the two sets of vectors

$$\left\{\frac{\mathbf{0}}{r_0},\frac{\mathbf{v}_i}{r_i},\quad i=1,\dots,n+1\right\},$$
$$\left\{\frac{\mathbf{0}}{-r_0},\frac{\mathbf{v}_i}{r_i-r_0},\quad i=1,\dots,n+1\right\} \tag{161}$$

are both in generic position in R^n, since they are obtained by applying an arbitrarily small perturbation to the set of vectors (158), which are by hypothesis generic in R^n.

VI. MAXIMUM NUMBER OF INDEPENDENT SYMMETRIES OF THE SYSTEM $x_i^{(r_i}=F_i(t;x_1,\dots,x_1^{(r_1-1};\dots;x_n,\dots,x_n^{(r_n-1}),\ r_i>1$

The results obtained in Secs. II–V indicate that systems of differential equations of the form

$$\mathbf{x}^{(r}=\mathbf{F}(t,\mathbf{x},\dots,\mathbf{x}^{(r-1}),$$
$$\mathbf{x}=(x_1,\dots,x_n),\quad r>1, \tag{162}$$

possess a finite number of independent symmetry vectors and that the system $\mathbf{x}^{(r}=0$ possesses a number of independent symmetries that tends to infinity when either r or n tend to infinity, thus showing that the upper bound for the maximum number of independent symmetry vectors of (162) tends to infinity when either r or n tend to infinity.

We shall see in this section that these results hold as well for the more general class of systems of the form

$$x_i^{(r_i}=F_i(t;\mathbf{y}),$$
$$\mathbf{y}=(x_1,\dots,x_1^{(r_1-1};\dots;x_n,\dots,x_n^{(r_n-1}), \tag{163}$$
$$i=1,\dots,n,\quad 1<r_i.$$

The restriction $r_i>1$ for every i is essential for the validity of these results, since it is not difficult to give examples of systems of the form (163) with $r_{i_0}=1$ for some i_0 possessing an infinite number of independent symmetries. This is what happens, for example, with "split" systems of the form

$$\dot x_1=F_1(t,x_1),$$
$$x_i^{(r_i}=F_i(t;x_2,\dots,x_2^{(r_2-1};\dots;x_n,\dots,x_n^{(r_n-1}), \tag{164}$$
$$r_i>1\quad\text{for every } i=2,\dots,n,$$

admitting an infinite number of independent symmetries of the form

$$t'=t,\quad x_1'=x_1+\epsilon\cdot\psi(t,x_1),$$
$$x_i'=x_i\quad\text{for every } i=2,\dots,n, \tag{165}$$

where $\psi(t,x_1)$ is such that $\psi(t,x_1)\partial/\partial x_1$ is a symmetry vector of the equation

$$\dot x_1=F_1(t,x_1) \tag{166}$$

[since it is well known that every first-order equation like (166) admits an infinite number of independent symmetries[10]].

A less trivial example of a differential system of the form (163) with $r_{i_0}=1$ for some i_0 admitting an infinite number of independent symmetries is the following:

$$\dot x=F(t,x),\quad \ddot y=G(t,x). \tag{167}$$

Indeed, the necessary and sufficient condition in order that $\mathbf{S}(t,x,y)=\eta(t,x)\,\partial/\partial y$ be a symmetry vector of (167) turns out to be the following linear partial differential equation in η:

$$\eta_{tt}=-2\eta_{tx}F-\eta_{xx}F^2-\eta_x\dot F\quad(\dot F=F_t+F_xF). \tag{168}$$

Equation (168) is Kowalewskian in the variable t, and therefore[11] possesses an infinite number of local solutions, depending on two arbitrary functions $f(x)$ and $g(x)$; for instance;

$$f(x)=\eta(0,x),\quad g(x)=\eta_t(0,x). \tag{169}$$

Therefore, the system (167) possesses an infinite number of independent symmetries, as claimed.

(a) We begin now the proof of the assertions made at the beginning of this section.

As in previous sections [II(a), (b), (c); III(a)] it is not difficult to show that, given the initial value $(t_0,\mathbf{y}_0)$, where

$$\mathbf{y}_0=(x_{01},\dots,x_{01}^{(r_1-1};\dots;x_{0n},\dots,x_{0n}^{(r_n-1}), \tag{170}$$

and denoting by $\boldsymbol\phi(t)$ the solution of (163) corresponding to this initial condition, for $P_1=(t_1,\boldsymbol\phi(t_1))=(t_1,\mathbf{x}_1)$ sufficiently close to $P_0=(t_0,x_{01},\dots,x_{0n})=(t_0,\mathbf{x}_0)$ one can find an open neighborhood U in R^{1+n} such that through every point P of U there pass two integral curves of (163), $\gamma_1=\{(t,\boldsymbol\phi_1(t)\,|\,t\in I_1\subset R\}$ and $\gamma_2=\{(t_1,\boldsymbol\phi_2(t))\,|\,t\in I_2\subset R\}$, satisfying

$$\phi_{1i}^{(k}(t_0)=x_{0i}^{(k},\quad \phi_{2i}^{(k}(t_1)=x_{1i}^{(k} \tag{171a}$$
for every $k=1,\dots,r_i-2$ and $i=1,\dots,n$,
$$P\text{ is isolated in }\gamma_1\cap\gamma_2, \tag{171b}$$

where we have set $x_{1i}^{(k}=\phi_i^{(k}(t_1)$.

(b) Let

$$\mathbf{S}=\varphi(t,\mathbf{x})\frac{\partial}{\partial t}+\sum_{i=1}^{n}\psi_i(t,\mathbf{x})\frac{\partial}{\partial x_i} \tag{172}$$

be a symmetry vector of (163). If for every $P\in U$ the two integral curves of (163), γ_1 and γ_2, defined above are invariant under $\mathbf{S}$, then P will be invariant under $\mathbf{S}$, since by (171b) P is isolated in $\gamma_1\cap\gamma_2$, and, consequently, $\mathbf{S}$ will vanish at P for every $P\in U$, i.e., $\mathbf{S}$ will vanish identically on U. Therefore, by the arguments given in Sec. II(c), (e), to compute an upper bound for the maximum number of independent symmetry vectors of (163) it suffices to find the number of linear equations in the components of $\mathbf{S}$ that guarantee the invariance of the following linear elements of order (r_n-1) under S^{r_n-1}:

$$\begin{aligned}\mathbf{z}_0&=(t_0,x_{01},\ldots,x_{01}^{(r_1-2},\xi_1,F_{01},\ldots,F_{01}^{(r_n-r_1};\ldots;\\&\quad x_{0n},\ldots,x_{0n}^{(r_n-2},\xi_n),\\ \mathbf{z}_1&=(t_1,x_{11},\ldots,x_{11}^{(r_1-2},\xi_1,F_{11},\ldots,F_{11}^{(r_n-r_1};\ldots;\\&\quad x_{1n},\ldots,x_{1n}^{(r_n-2},\xi_n)\end{aligned} \tag{173}$$

for *every* $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)$, $\quad r_n=\max_i r_i$,

where we have set

$$F_{\sigma i}^{(k}=\frac{d^k}{dt^k}F_i\Big|_{(t,x_{\sigma 1}^{(r_1-2},\ldots,x_{\sigma 1}^{(r_1-2},\xi_1;\ldots;x_{\sigma n},\ldots,x_{\sigma n}^{(r_n-2},\xi_n)},$$

$$\sigma=0,1,$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\Big[\dot{x}_i\frac{\partial}{\partial x_i}+\cdots+x_i^{(r_i-1}\frac{\partial}{\partial x_i^{(r_i-2}}+F_i\frac{\partial}{\partial x_i^{(r_i-1}}\Big]. \tag{174}$$

[Equations (174) simply state that the derivatives of $\mathbf{F}$ appearing in (173) are to be computed along the integral curves of (163), γ_1—for $F_{0i}^{(k}$—and γ_2—for $F_{1i}^{(k}$.]

The invariance of the linear elements (173) for *every value* of $\boldsymbol{\xi}$ is in turn equivalent to the following set of linear equations in the functions φ and ψ_i:

$$\begin{aligned}&\forall\boldsymbol{\xi},\quad \varphi(P_0)=\psi_i(P_0)=\psi_i^k(\mathbf{z}_0)=0,\\&k=1,\ldots,r_n-1,\quad i=1,\ldots,n,\end{aligned} \tag{175}$$

for $\mathbf{z}_0$, and

$$\begin{aligned}&\forall\boldsymbol{\xi},\quad \varphi(P_1)=\psi_i(P_1)=\psi_i^k(\mathbf{z}_1)=0\\&k=1,\ldots,r_n-1,\quad i=1,\ldots,n\end{aligned} \tag{176}$$

for $\mathbf{z}_1$. At this point it is important to note that $\psi_i^k(\mathbf{z}_\sigma)$ depends on $\boldsymbol{\xi}$ not only explicitly, but also implicitly, through $F_{\sigma j}^{(p}$ $(p=1,\ldots,r_n-r_j, j=1,\ldots,n)$.

More precisely, taking into account the structure of ψ_i^k, given by (79) and (94), we see that $\psi_i^k(\mathbf{z}_\sigma)$ depends on the variables $\boldsymbol{\xi}$, $F_{\sigma j}^{(p}$ *polynomially*; $\psi_i^k(\mathbf{z}_\sigma)$ is a polynomial in the variables $\boldsymbol{\xi}$, $F_{\sigma j}^{(p}$ whose coefficients are linear combinations of the partial derivatives of the functions φ and $\boldsymbol{\psi}$ evaluated at P_σ. The *constant* values of $(t_\sigma;x_{\sigma 1},\ldots,x_{\sigma 1}^{(r_1-2};\ldots;x_{\sigma n},\ldots,x_{\sigma n}^{(r_n-2})$ appear in $\psi_i^k(\mathbf{z}_\sigma)$ implicitly through the variables $F_{\sigma j}^{(p}$ and explicitly as coefficients of the partial derivatives of the functions φ and $\boldsymbol{\psi}$.

As a consequence, it immediately follows that the fulfillment of (175)–(176) is guaranteed by a *finite* number of *linear* conditions on the functions φ and $\boldsymbol{\psi}$, namely the vanishing of $\varphi(P_\sigma)$, $\boldsymbol{\psi}(P_\sigma)$ and of all the coefficients appearing in $\psi_i^k(\mathbf{z}_\sigma)$ regarded as a polynomial in the variables $\boldsymbol{\xi}$ and $F_{\sigma j}^{(p}$ $(p=1,\ldots,r_n-r_j; j=1,\ldots,n)$, for every value of $k=1,\ldots,r_n-1$, $i=1,\ldots,n$ and for $\sigma=0,1$. Indeed, these conditions involve only the constant values of $(t_\sigma,\mathbf{x}_\sigma;\dot{x}_{\sigma 1},\ldots,x_{\sigma 1}^{(r_1-2};\ldots;\dot{x}_{\sigma n},\ldots,x_{\sigma n}^{(r_n-2})$, and their linearity in φ, $\boldsymbol{\psi}$ is a direct consequence of the linearity of (175), (176) in φ and $\boldsymbol{\psi}$.

Clearly, not all of the above conditions are independent: For instance, following the reasoning of Sec. III(c), it would be easy to verify that the vanishing of $\boldsymbol{\psi}(P_1)$ and of the term independent of the variables $\boldsymbol{\xi}$, $F_{1j}^{(p}$ in $\psi_i^k(\mathbf{z}_1)$ are a consequence of all the other conditions, and therefore this condition could be omitted. But the point here is that, at any rate, the number of the conditions obtained above is *finite*; therefore, the argument given in Sec. II(f) shows that the number of independent symmetry vectors of (163) is also finite, since it cannot exceed the number of these conditions.

(b) We shall now see that the least upper bound on the number of independent symmetries of (163) tends to infinity when either n or some of the r_i tend to infinity. Indeed, consider the system

$$x_i^{(r_i}=0,\quad i=1,\ldots,n. \tag{177}$$

The necessary and sufficient condition in order that (172) be a symmetry vector of (177) can be expressed as follows:

$$\psi_i^{r_i}\big|_{x_k^{(r_k}=0,\ k=1,\ldots,n}=0,\quad i=1,\ldots,n. \tag{178}$$

Taking into account the structure of $\psi_i^{r_i}$, given by Eq. (79), we observe that (177) admits the particular solutions

$$\varphi=0,\quad \frac{d^{r_i}\psi_i}{dt^{r_i}}\Big|_{x_k^{(r_k}=0,\ k=1,\ldots,n}=0. \tag{179}$$

A particular solution of (179) is the following one, dependent on $r_1+r_2+\cdots+r_n$ arbitrary constants:

$$\begin{aligned}&\varphi=0,\quad \psi_i=a_i^0+a_i^1t+\cdots+a_i^{r_i-1}t^{r_i-1},\\&i=1,\ldots,n.\end{aligned} \tag{180}$$

From (180) we obtain the following set of $r_1+r_2+\cdots+r_n$ independent symmetry vectors of (177):

$$\frac{\partial}{\partial x_i};\quad t\frac{\partial}{\partial x_i};\quad \cdots;\quad t^{r_i-1}\frac{\partial}{\partial x_i}. \tag{181}$$

Since the number $r_1+r_2+\cdots+r_n$ evidently tends to infinity when either n or some of the r_i tend to infinity, it follows that the same thing will happen with the least upper bound on the number of independent symmetries of (163), since the least upper bound by definition is greater than or equal to the number of independent symmetries of (177), which in turn exceeds the number $r_1+r_2+\cdots+r_n$, as we have just shown.

VII. FINAL REMARKS

It has been shown that a system of differential equations of the type (163) can only admit an ordinary local Lie group (i.e., a local Lie group with a *finite* number of essential parameters) of pointlike symmetries. This result precludes the possibility that a system of differential equations of this kind admit a Lie group of symmetries with an infinite number of parameters (as the formal group of locally invertible transformations of the manifold $\{(t,\mathbf{x})\} = R^{1+n}$, for instance). As is well known, this result is no longer valid when dynamical symmetries are considered (see, e.g., the paper by the authors cited in Ref. 1).

It has also been shown that a system of differential equations of the kind (70), with $r > 2$, does not admit more than $N(r,n)$ independent symmetries, where the number $N(r,n)$ satisfies the following inequalities:

$$n^2 + nr + 3 \leqslant N(r,n) \leqslant 2n^2 + nr + 2. \tag{182}$$

In addition, the system $\mathbf{x}^{(r} = \mathbf{0}$ has exactly $n^2 + nr + 3$ independent symmetries: Therefore, it would be nice to show that, when $n > 1$, this number cannot be surpassed by the number of independent symmetries of any system of the kind (70), or, if this were not the case, to exhibit a system of this kind having more than $n^2 + nr + 3$ independent symmetries. Also open is the problem of obtaining computational algorithms for constructing systems of the kind (70) with any preassigned number of symmetries s [not exceeding the maximum number of independent symmetries allowed to every equation of the kind (70), for given n and r].

When $n = 1$, the least upper bound to the number of independent symmetries of of an equation of the kind (1) when $r > 2$ is given by the number $r + 4$, this number being equal to the number of independent symmetries of the equation $x^{(r} = 0$ when $r > 2$.

If $r = 2$, the least upper bound to the number of independent symmetries of (70) is given by $n^2 + 4n + 3$, the number of independent symmetries of the system $\ddot{\mathbf{x}} = \mathbf{0}$. Therefore, in this case no new feature distinguishes the two cases $n > 1$ and $n = 1$, since in both cases the maximum number of symmetries is attained by the system (or equation) $\ddot{\mathbf{x}} = \mathbf{0}$ ($\ddot{x} = 0$).

It is also interesting to notice that the least upper bound to the number of independent symmetries of a system of the kind (163) tends to infinity when either n or some of the r_i tend to infinity; this result is not completely unexpected, in view of the fact that the general solution of (163) depends on $r_1 + r_2 + \cdots + r_n$ parameters.

Another interesting consequence of the previous results is that, when r is kept fixed—say $r = 2$, which is the case of Newtonian mechanics—and a certain group G of transformations of the manifold $\{(t,x_1,...,x_n)\}$ depending on s parameters is given, then no equation of the form $\ddot{\mathbf{x}} = F(t,\mathbf{x},\dot{\mathbf{x}})$ can possess as many symmetries as G if $s > n^2 + 4n + 3$. But considering the action of G^N, the group of transformations of the manifold $\underbrace{R^{1+n} \times \cdots \times R^{1+n}}_{N}$ induced by G, the possibility remains open that, for N sufficiently high, the group G^N, which also possesses s essential parameters, is a symmetry group (of generally nonpointlike transformations) of some system of the form

$$\ddot{\mathbf{x}}_i = \mathbf{F}_i(t,\mathbf{x}_1,...,\mathbf{x}_N,\dot{\mathbf{x}}_1,...,\dot{\mathbf{x}}_N)$$
$$i = 1,...,N, \quad N > 1 \quad [x_i = (x_1,...,x_{ni})]. \tag{183}$$

If this were the case for any G, we could then assert that any group of pointlike transformations of the manifold $\{(t,\mathbf{x})\} = R^{1+n}$ could be considered, when extended in the natural way to systems of more than one Newtonian particle, as a symmetry group of a system of this kind. The problem would be, of course, to find the number N appropriate for a given group G and, more importantly, the functions $\mathbf{F}_i$ appearing in (183).

Further work on these open problems is going on and will appear in forthcoming papers of this series.

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APPENDIX

For completeness reasons, we give here some definitions concerning the projective group and a direct proof showing that this group is a symmetry group of the system $\ddot{\mathbf{x}} = \mathbf{0}$.

(i) Real $(m-1)$-dimensional projective space RP^{m-1} is usually defined as the quotient set

$$RP^{m-1} = (R^m - \{0\})/\sim \tag{A1}$$

where $\sim$ denotes the following equivalence relation:

$$\mathbf{y} \sim \mathbf{x} \Leftrightarrow \mathbf{y} = c\mathbf{x},$$
$$\mathbf{x},\mathbf{y} \in R^m - \{0\}, \quad c \in R - \{0\}. \tag{A2}$$

Therefore, the elements of RP^{m-1} are straight lines passing through the origin, with the origin removed. It is a standard result[12] that RP^{m-1} is a differentiable manifold, with the differentiable structure induced by the charts (U_i, φ_i) defined by

$$U_i = \{[\mathbf{x}] \in RP^{m-1} | x_i \neq 0\},$$
$$\varphi_i([\mathbf{x}]) = (x_1/x_i,...,x_{i-1}/x_i, x_{i+1}/x_i,...,x_m/x_i), \tag{A3}$$
$$i = 1,...,m,$$

where $[\mathbf{x}]$ denotes the equivalence class of $\mathbf{x} \in R^m - \{0\}$.

Geometrically, $(x_1/x_i,...,x_{i-1}/x_i, 1, x_{i+1}/x_i,...,x_m/x_i)$ are nothing but the coordinates of the point of R^m defined by the intersection of the straight line $[\mathbf{x}]$ with the hyperplane $x_i = 1$.

Every linear nonsingular transformation $L: R^m \to R^m$ canonically induces a so-called projective transformation $\hat{L}$: $RP^{m-1} \to RP^{m-1}$ as follows:

$$\hat{L}([\mathbf{x}]) = [L\mathbf{x}] \quad \forall [\mathbf{x}] \in RP^{m-1}. \tag{A4}$$

$\hat{L}$ is well defined, since L is by hypothesis nonsingular and therefore $\mathbf{x} \in (R^m - \{0\}) \Leftrightarrow L\mathbf{x} \in (R^m - \{0\})$.

Geometrically, $\hat{L}([\mathbf{x}])$ is nothing but the straight line (with the origin removed) obtained by transforming the straight line $[\mathbf{x}]$ under L.

Let us see now what is the expression of a projective transformation in terms of the coordinates of one of the charts (A3), for instance, the chart (U_m, φ_m). If we denote by

l_{ij} $(i,j = 1,...,m)$ the matrix elements of L relative to the canonical basis of R^m, then we have

$$U_m \cap \hat{L}^{-1}(U_m) = V_m, \tag{A5}$$

where V_m is the open subset of U_m (and hence of RP^{m-1}, since U_m is itself open in RP^{m-1}) defined by

$$V_m = \{[\mathbf{x}] \in RP^{m-1} \,|\, \mathbf{x} \notin \Pi\} \cap U_m, \tag{A6}$$

Π being the hyperplane of R^m whose equation is

$$\Pi: \quad \sum_{i=1}^{m} l_{mi} x_i = 0. \tag{A7}$$

If we denote by

$$u_i = x_i / x_m, \quad i = 1,...,m-1, \tag{A8}$$

the coordinates of $[\mathbf{x}] \in V_m$ relative to the chart (U_m, φ_m), then the coordinates of $\hat{L}([\mathbf{x}]) \in U_m$ relative to the same chart will be given by

$$u_i' = \frac{(L\mathbf{x})_i}{(L\mathbf{x})_m} = \frac{\sum_{j=1}^{m} l_{ij} x_j}{\sum_{j=1}^{m} l_{mj} x_j} = \frac{\sum_{j=1}^{m} l_{ij} u_j + l_{im}}{\sum_{j=1}^{m} l_{mj} u_j + l_{mm}}, \quad i = 1,...,m-1. \tag{A9}$$

Note that $\hat{L}$ depends on $m^2 - 1$ essential parameters since, for any $c \neq 0$, L and cL induce the same projective transformation $\hat{L}$.

From the identities

$$\hat{L}_1 \hat{L}_2 = (L_1 L_2)^{\wedge}, \quad (\hat{L})^{-1} = (L^{-1})^{\wedge} \tag{A10}$$

it follows that the set of all projective transformations forms a group, called the *projective group*: the dimension of the projective group of RP^m is, according to what has been said above, equal to $(m+1)^2 - 1 = m^2 + 2m$.

(ii) Let us show now that the system of differential equations

$$\ddot{\mathbf{x}} = \mathbf{0}, \quad \mathbf{x} = (x_1,...,x_n) \tag{A11}$$

is symmetrical under the local transformations (sufficiently close to the identity) defined by

$$x_i' = \frac{\sum_{j=1}^{n+1} l_{ij} x_j + l_{i,n+2}}{\sum_{j=1}^{n+1} l_{n+2,j} x_j + l_{n+2,n+2}} \quad \text{with } x_{n+1} = t, \tag{A12}$$

where it is understood that the point $(x_1,...,x_n,t)$ belongs to a certain open subset W of R^{n+1} such that the denominator appearing in (A12) does not vanish on W.

We can regard $(x_1,...,x_n,t)$ as the coordinates relative to the chart (U_{n+2}, φ_{n+2}) of the point $[\mathbf{y}] \in RP^{n+1}$ defined as follows:

$$[\mathbf{y}] = [(x_1,...,x_n,t,1)]. \tag{A13}$$

Similarly, we consider (A12) as the expression in the chart (U_{n+2}, φ_{n+2}) of the projective transformation $\hat{L}$ induced by the linear transformation $L: R^{n+2} \to R^{n+2}$ whose matrix elements (relative to the canonical basis of R^{n+2}) are the numbers l_{ij} $(i,j = 1,...,n+2)$ appearing in (A12).

The general solution of the system (A11) is the following:

$$x_i(t) = a_i t + b_i, \quad i = 1,...,n, \; a_i, b_i \in R, \tag{A14}$$

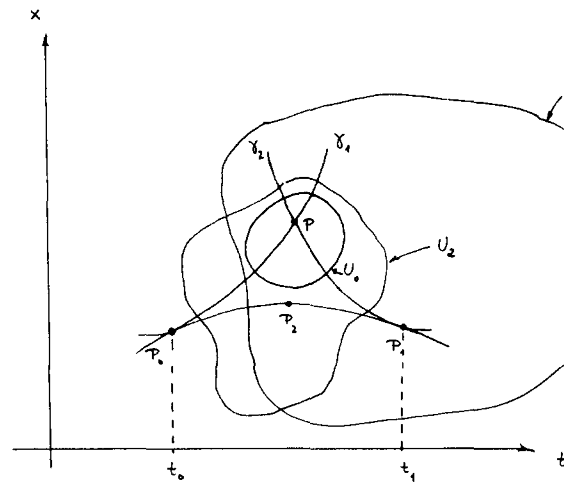

FIG. 1

which can be regarded as the implicit equation of the straight line of RP^{n+1} whose parametric equations are

$$x_i = a_i \lambda + b_i, \quad x_{n+1} = \lambda, \quad i = 1,...,n, \; \lambda \in \mathbb{R}. \tag{A15}$$

To the "straight line" (A15) there corresponds the following subset of R^{n+2}:

$$y_i = \mu(a_i \lambda + b_i), \quad y_{n+1} = \mu\lambda, \quad y_{n+2} = \mu, \quad i = 1,...,n, \quad \lambda \in R, \mu \in R - \{0\}. \tag{A16}$$

Geometrically, (A16) is obtained from (A15) as follows: for each point of the form $(a_1\lambda + b_1,...,a_n\lambda + b_n,t,1)$ in the hyperplane $y_{n+2} = 1$ of R^{n+2}, we draw the straight line joining this point to the origin of R^{n+2}; the union of all the straight lines thus obtained with the origin removed is precisely the subset of R^{n+2} defined by (A16).

It is not difficult to verify that (A16) can be alternatively obtained from the two-dimensional subspace Π_2 or R^{n+2} defined by

$$y_i = \nu a_i + \mu b_i, \quad i = 1,...,n, \quad y_{n+1} = \nu, \quad y_{n+2} = \mu, \quad \nu, \mu \in R, \tag{A17}$$

by simply removing all the points of the straight line $r \subset \Pi_2$ given by

$$y_i = \eta a_i, \quad i = 1,...,n, \quad y_{n+1} = \eta, \quad y_{n+2} = 0, \quad \eta \in R \tag{A18}$$

(see Fig. 2).

Since L is a linear, nonsingular transformation, it transforms $\Pi_2 - r$ into $\Pi_2' - r'$, where $\Pi_2' = L(\Pi_2)$ is a two-dimensional subspace of R^{n+2} and $r' = L(r)$ is a straight line contained in Π_2'. Furthermore, since (A12) can be chosen arbitrarily close to the identity (whose parameters are given by $l_{ij} = c\delta_{ij}$, for every $c \in R - \{0\}$), it follows that $\Pi_2' - r'$ intersects the hyperplane $y_{n+2} = 1$, since this hyperplane intersects the set $\Pi_2 - r$.

The intersection of $\Pi_2' - r'$ with the hyperplane $y_{n+2} = 1$ is a straight line, whose equation we write in the form

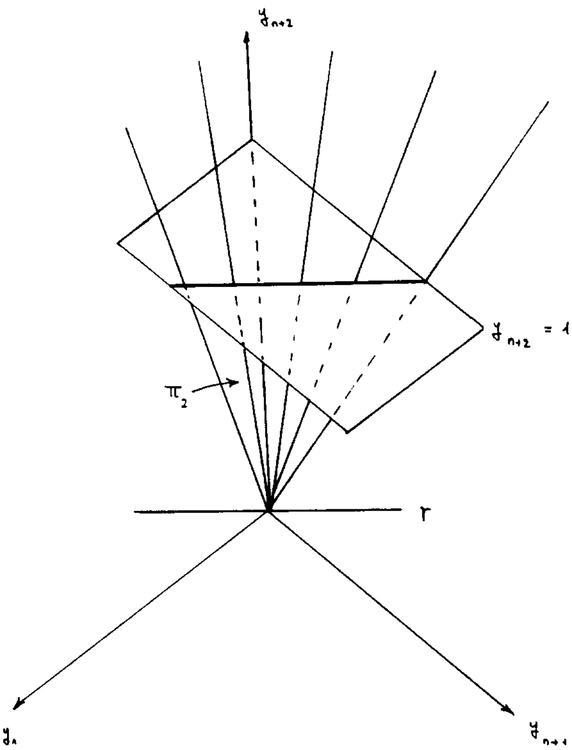

FIG. 2.

$$y_i = \alpha_i \mu + \beta_i, \quad y_{n+2} = 1, \qquad i = 1,\dots,n+1, \quad \mu \in R, \tag{A19}$$

where α_i and β_i are fixed real numbers depending on a_j and b_j $(j = 1,\dots,n)$ and on the matrix elements l_{ij} $(i,j = 1,\dots,n+2)$ of L.

It follows that the equations in the chart (U_{n+2},φ_{n+2}) of the subset of RP^{n+1} obtained by applying $\hat{L}$ to the subset of RP^{n+1} whose equation—in the same chart—is (A15) are the following:

$$x_i' = \alpha_i \mu + \beta_i, \quad i = 1,\dots,n+1. \tag{A20}$$

Since $\hat{L}$ is arbitrarily close to the identity, and clearly $\alpha_{n+1} = 1$ when $\hat{L}$ is equal to the identity, it follows that $\alpha_{n+1} \neq 0$; therefore, we can use the $(n+1)$th equation of (A20) to solve for μ as a function of t':

$$\mu = (t' - \beta_{n+1})/\alpha_{n+1}. \tag{A21}$$

Substituting back into (A20), we see that (A20) is equivalent to the following set of equations:

$$x_i' = A_i t' + B_i, \quad i = 1,\dots,n, \tag{A22}$$

where

$$A_i = \frac{\alpha_i}{\alpha_{n+1}}, \quad B_i = \frac{\alpha_{n+1}\beta_i - \alpha_i \beta_{n+1}}{\alpha_{n+1}}. \tag{A23}$$

Since we have shown that by applying (A12) to an arbitrary solution (A14) of (A11) we obtain another solution of (A11), given by (A22), it follows that (A12) is a symmetry of the system (A11), as we had claimed.

ADDENDUM

We shall show here that the function $R(t,\lambda)$ defined by (8) is a C^∞ function.

Indeed, we have

$$(t-t_0)^r R(t,\lambda) = \frac{1}{(r-1)!}\int_{t_0}^{t} \phi^{(r}(s,\lambda)(t-s)^{(r-1} \, ds \tag{i}$$

by Cauchy's integral form for the remainder of the Taylor expansion of $\phi(t,\lambda)$ around $t = t_0$ (for fixed λ).[13]

Performing the change of variable,

$$s = t_0 + u(t-t_0), \quad u \in [0,1], \tag{ii}$$

we immediately obtain

$$\int_{t_0}^{t} \phi^{(r}(s,\lambda)(t-s)^{(r-1} \, ds = (t-t_0)^r \int_0^1 \phi^{(r}(t_0 + u(t-t_0),\lambda)(1-u)^{r-1} \, du. \tag{iii}$$

Comparing (i) with (iii), we get

$$R(t,\lambda) = \frac{1}{(r-1)!}\int_0^1 \phi^{(r}(t_0 + u(t-t_0),\lambda) \times (1-u)^{r-1} \, du, \tag{iv}$$

which is clearly of class C^k in the variables (t,λ) provided that $\phi(t,\lambda)$ is of class C^{r+k}.

Since by hypothesis $\phi(t,\lambda)$ is of class C^∞, it follows that $R(t,\lambda)$ is also of class C^∞, as claimed.

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